Co-analytic mad families, large continuum and global Σ-uniformization

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Abstract

We construct a model in which $\mathfrak{b} = \mathfrak{c} = \aleph_3$, there is a Δ^1_3-definable well-order of the reals, there is a Π^1_1 mad family, and, for every $n \geq 2$, every Σ^1_n subset of the plane has a Σ^1_n uniformizing function.

1 Introduction

Uniformization is the problem of choosing witnesses in a definable way. If $A \subseteq X \times Y$, a uniformization of A is a set $U \subseteq A$ which is the graph of a function and satisfies

$$\mathrm{dom}(U) = \mathrm{proj}_X(A).$$

Thus U chooses exactly one y from each non-empty vertical section $A_x = \{y : (x, y) \in A\}$. Uniformization theorems are used throughout classical and effective descriptive set theory, and also in applications such as ergodic theory and the study of definable classification problems. Kondo's uniformization theorem gives, in ZFC, uniformization for co-analytic sets; in particular it implies Σ^1_2-uniformization by uniformizing a Π^1_1 relation in the variables $(x, (y, z))$ and then projecting away the auxiliary coordinate z [Kec95]. This is the sharp ZFC boundary relevant here: beyond this level, full projective uniformization is not provable in ZFC. For a projective pointclass Γ, let Γ-UP denote the assertion that every $A \subseteq 2^\omega \times 2^\omega$ in Γ has a uniformization whose graph is again in Γ. We say that global Σ-uniformization holds if Σ^1_n-UP holds for every $n \geq 2$.

Historically, the standard route to higher projective uniformization went through good projective well-orders, following Addison's theorem [Add59]. We use the standard formulation recorded by Kanamori. Suppose $\prec$ is a strict well-ordering of ω^ω of order type ω_1, and let $IS_\prec$ be the relation which says that the first coordinate enumerates the proper $\prec$-initial segment of the second coordinate:

$$IS_\prec(s, y) \iff \{(s)_i \mid i < \omega\} = \{z \in \omega^\omega \mid z \prec y\},$$

where s is decoded as a sequence of reals. For a pointclass Γ, the well-order $\prec$ is Γ-good if $IS_\prec \in \Delta\Gamma$, where $\Delta\Gamma = \Gamma \cap \neg\Gamma$ [Kan09, 13.11, 29.10]. Thus goodness is stronger than merely being a projective well-order with countable initial segments: the relation which presents those initial segments by reals must itself have the required projective complexity. This is precisely the feature that lets one express "there is no earlier witness" without leaving the relevant pointclass, and Addison's least-witness argument then gives uniformization by choosing, in each non-empty section, the $\prec$-least witness.

This classical route makes global Σ-uniformization look fragile: it ties the construction to a low-complexity good well-order and, in that form, to CH. Recent forcing constructions provide a different route and show that this impression is too pessimistic [Hof25a, Hof25b]. The present paper uses this newer coding method to test the flexibility of global Σ-uniformization under an additional combinatorial requirement on the real line. We integrate the global Σ-coding with the preservation of a co-analytic mad family and with the large-continuum pattern $\mathfrak{b} = \mathfrak{c} = \aleph_3$. Thus global Σ-uniformization can coexist with a co-analytic maximal almost disjoint family in a model with $\mathfrak{b} = \mathfrak{c} = \aleph_3$. Since $\mathfrak{b} \leq \mathfrak{a}$, the mad family in the final model is necessarily large; it is not a small ground-model object accidentally surviving the forcing, but the reinterpretation of a definable $\aleph_1$-union of perfect almost disjoint sets in a universe with continuum $\aleph_3$.

We recall some background on mad families. A family $\mathcal{A} \subseteq [\omega]^\omega$ is almost disjoint if $a \cap b$ is finite whenever $a, b \in \mathcal{A}$ are distinct. It is maximal almost disjoint, or mad, if every infinite subset of ω has infinite intersection with some member of $\mathcal{A}$. Mathias [Mat77] proved that there are no analytic mad families, while Miller [Mil89] proved that L contains a Π^1_1 mad family. Törnquist [Tör13] later proved that the existence of a Σ^1_2 mad family already implies the existence of a Π^1_1 mad family; since the converse is immediate, the two existence statements are equivalent. The proof of the non-trivial implication uses Kondo's uniformization theorem for co-analytic sets. Starting from a Σ^1_2 definition of a mad family, the relevant choices are organized as a Π^1_1 relation, and Kondo uniformization supplies a co-analytic choice relation from which one extracts a Π^1_1 mad family. Thus uniformization already appears as a tool for lowering the projective complexity of definable mad families. The present paper develops the connection in a different direction: we preserve a co-analytic mad family while forcing global Σ-uniformization. Brendle and Khomskii [BK13] introduced the preservation method used to build $\aleph_1$-perfect mad families which remain maximal after suitable iterations adding dominating reals. Fischer, Friedman and Khomskii [FFK13] combined this method with the Friedman–Zdomskyy and Fischer–Friedman–Zdomskyy coding machinery [FZ10, FFZ11] and obtained a model with

$$\mathfrak{b} = \mathfrak{c} = \aleph_3,$$

a Δ^1_3 well-order of the reals, and a Π^1_1 mad family.

Our construction uses the Fischer–Friedman–Khomskii preparation, but the role of that preparation is broader than in the original theorem. It supplies the localized coding infrastructure, the definable diamond sequence needed for the Brendle–Khomskii mad family, and the Laver-style almost-disjoint coding forcings which are σ-centered and strongly preserve splitting reals. We then run a new combined iteration over this preparation. The same exact localized coding predicate $\mathrm{Coded}(z)$ serves simultaneously as the predicate from which the Δ^1_3 well-order is read and as the base Σ^1_3 predicate used in all higher uniformization arguments. We prove the following theorem:

Theorem 1.1. *It is consistent with* ZFC *that the following statements hold simultaneously:*

(1) $\mathfrak{b} = \mathfrak{c} = \aleph_3$;

(2) *there is a* Δ^1_3*-definable well-order of the reals;*

(3) *there is a* Π^1_1 *mad family;*

(4) *for every* $n \geq 2$*, the pointclass* Σ^1_n *has the uniformization property.*

The proof is written for $\aleph_3$. The choice of $\aleph_3$ is not particularly important, the proof would work with a more complicated, but straightforward generalization of the preparatory models L^* and $L^\dagger$ for any $\aleph_n$, for n an integer ≥ 2. For even bigger values of the continuum one needs different coding mechanisms which are beyond the scope of this article.

The paper is organized as follows. Section 2 recalls the part of the Fischer–Friedman–Khomskii preparation needed for the argument and formulates the strong preservation theorem for the $\aleph_1$-perfect mad family. Section 3 isolates the localized Σ^1_3 coding predicate and its exactness properties. Section 4 defines the combined forcing, including the well-order stages, the uniformization stages, the dummy prelude, and the coherent compression maps. Section 5 verifies the final model: the values of $\mathfrak{b}$ and $\mathfrak{c}$, the Δ^1_3 well-order, the Π^1_1 mad family, and the global Σ-uniformization property. The final section records the dependence of the stated theorem on the current length-ω_3 preparation.

2 The Fischer–Friedman–Khomskii preparation

This section records the part of the Fischer–Friedman–Khomskii construction which will be used in the combined model. The point is twofold. First, their preparatory forcing produces a model L^* in which the Friedman–Zdomskyy local coding apparatus is available and in which there is still a sufficiently definable diamond sequence. Secondly, the final coding forcings may be chosen so that they strongly preserve splitting reals. This is exactly the preservation property needed to keep the Brendle–Khomskii perfect mad family maximal through the long finite-support iteration.

2.1 The preliminary extension

We start in L. Recall that a transitive model M of a sufficiently large fragment of ZF is called *suitable* if ω_3^M exists and $\omega_3^M = \omega_3^{L^M}$. Then also $\omega_1^M = \omega_1^{L^M}$ and $\omega_2^M = \omega_2^{L^M}$. The preliminary forcing of Fischer–Friedman–Khomskii is a three-step forcing

$$\mathbb{P}^* = \mathbb{P}^0 * \dot{\mathbb{P}}^1 * \dot{\mathbb{P}}^2.$$

Only the properties of this forcing will be used later, but we recall the structure in order to make clear where the later projective definitions come from.

In the first step one fixes, in L, a $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1))$-sequence and obtains a mutually almost disjoint sequence

$$\vec{S} = \langle S_\alpha \mid 1 < \alpha < \omega_3 \rangle$$

of stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega_1)$. For $\omega_2 \leq \alpha < \omega_3$, the factor $\mathbb{P}^0_\alpha$ shoots a club through $\omega_2 \setminus S_\alpha$ by countable closed approximations; for $\alpha < \omega_2$ the factor is trivial. The product is taken with supports of size $\leq \omega_1$. The resulting forcing $\mathbb{P}^0$ is countably closed, ω_2-distributive and ω_3-c.c. In particular it adds no reals.

The second step codes the clubs added by $\mathbb{P}^0$ down to subsets of ω_1. For each $\alpha < \omega_3$ one first forms a set $D_\alpha \subseteq \omega_2$ which codes the relevant club C_α, the canonical L-code W_α for α, and the canonical code for the largest limit ordinal below or equal to α. One then codes D_α into a set $X_\alpha \subseteq \omega_1$ by an almost-disjoint coding over the standard diamond sequence on ω_1.

Let $\vartheta(u, v, X)$ be the fixed first-order decoding formula which says that, using the standard diamond almost-disjoint coding on u, the set $X \subseteq u$ decodes to a set $Z \subseteq v$ such that $\mathrm{Even}(Z)$ codes a triple $(C, W_{\bar{\alpha}}, W_{\bar{\gamma}})$, where $W_{\bar{\alpha}}$ and $W_{\bar{\gamma}}$ are the canonical L-codes for ordinals $\bar{\alpha}, \bar{\gamma} < \omega_3$, $\bar{\gamma}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and C is a club in v disjoint from $S_{\bar{\alpha}}$. The coding of D_α into X_α is arranged so that X_α satisfies the formula

$$\varphi(\alpha, X_\alpha) \iff \forall M\, \forall \beta \Big(\omega_1 < \beta \leq \omega_2 \ \wedge\ M \text{ is suitable } \wedge \omega_2^M = \beta \ \wedge\ \{X_\alpha\} \cup \omega_1 \subseteq M \implies M \models \vartheta(\omega_1, \omega_2, X_\alpha)\Big).$$

Thus the former local correctness condition is the assertion $\varphi(\alpha, X_\alpha)$. The product of these codings is countably closed and has the ω_2-c.c.; again no reals are added.

The third step localizes the information in the X_α's. For $\alpha \in \mathrm{Lim}(\omega_3)$ and $m < \omega$, the localization forcing adds a set $Y_{\alpha+m} \subseteq \omega_1$ whose initial segments carry enough information to reconstruct both $X_{\alpha+m}$ and X_α inside every relevant countable suitable model. This is expressed by the formula

$$\psi(\alpha, m, Y_{\alpha+m}) \iff \forall M\, \forall \beta \Big(\beta < \omega_1 \ \wedge\ M \text{ is suitable } \wedge \omega_1^M = \beta \ \wedge\ Y_{\alpha+m} \cap \beta \in M \implies M \models \vartheta(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \vartheta(\omega_1, \omega_2, X_\alpha \cap \beta)\Big).$$

The localization forcing at coordinate $\alpha+m$ adds $Y_{\alpha+m}$ so that $\psi(\alpha, m, Y_{\alpha+m})$ holds. The forcing $\mathbb{P}^2$ is the countable-support product of these localization forcings. It is ω-distributive in the relevant intermediate model and therefore does not add reals.

Let
$$L^* = L^{\mathbb{P}^*}.$$
We shall use the following package of consequences.

Lemma 2.1. *In L^* the following hold.*

(1) *No new reals have been added over L.*

(2) *Cardinals up to ω_3 are preserved,* CH *holds, and*
$$2^{\omega_1} = 2^{\omega_2} = \aleph_3.$$
Consequently
$$(\aleph_3)^{\aleph_0} = (\aleph_3)^{\aleph_1} = (\aleph_3)^{\aleph_2} = \aleph_3.$$

(3) *The sets X_α and $Y_{\alpha+m}$ satisfy the local correctness clauses described above.*

(4) *There is a Σ_1-definable $\diamondsuit'$-sequence over L_{ω_1}, hence a Σ_1-definable diamond sequence in the usual sense.*

Proof. The distributivity and chain condition computations are those of the Fischer–Friedman–Khomskii preliminary construction. The first forcing is countably closed, ω_2-distributive and ω_3-c.c. The second forcing is again countably closed and has the ω_2-c.c. The localization product is ω-distributive over the preceding extension. Hence the composition adds no reals, so $2^\omega = \aleph_1$ remains true.

We also record the cardinal arithmetic supplied by the preparation, since the compression maps in Section 4 use it explicitly. The size and chain-condition estimates from [FFK13] give the following name count. The full preparation has size $\aleph_3$ and the relevant antichains have size at most $\aleph_2$ (in fact smaller in the later ω_2-c.c. steps). Hence, for $\lambda \in \{\omega_1, \omega_2\}$, every subset of λ appearing after the preparation is represented by a nice name coded by a λ-sequence of subsets of $\mathbb{P}^*$ of size at most $\aleph_2$. Since L satisfies GCH,
$$\left((\aleph_3)^{\aleph_2}\right)^\lambda = \aleph_3 \qquad (\lambda = \aleph_1, \aleph_2),$$
there are at most $\aleph_3$ many such subsets. Thus $2^{\omega_1}, 2^{\omega_2} \leq \aleph_3$ in L^*.

The lower bound for 2^{ω_2} is inherited from L, where $2^{\omega_2} = \aleph_3$. The lower bound for 2^{ω_1} is supplied by the second step of the preliminary construction in [FFK13]: the sets $X_\alpha \subseteq \omega_1$ ($\alpha < \omega_3$) are pairwise distinct, because $\mathrm{Even}(X_\alpha)$ codes D_α, and D_α codes W_α, the L-least code for α. Therefore $2^{\omega_1} = 2^{\omega_2} = \aleph_3$ in L^*.

Finally,

$$(\aleph_3)^{\aleph_1} = (2^{\omega_1})^{\aleph_1} = 2^{\omega_1} = \aleph_3, \qquad (\aleph_3)^{\aleph_2} = (2^{\omega_2})^{\aleph_2} = 2^{\omega_2} = \aleph_3,$$

and hence also $(\aleph_3)^{\aleph_0} = \aleph_3$. Since the first two steps have the stated closure and chain condition properties, and the third step is designed not to add countable sequences of ordinals, the cardinals relevant to the subsequent ω_3-stage finite-support iteration are preserved.

The local correctness formulas $\varphi(\alpha, X_\alpha)$ and $\psi(\alpha, m, Y_{\alpha+m})$ are built into the definitions of Z_α, X_α and $Y_{\alpha+m}$. The point of the choice of Z_α is that whenever a suitable model sees a sufficiently long initial segment of the code, it belongs to the club E_α of correct elementary submodels and therefore computes the decoded triple correctly. The localization forcing then transfers this correctness to countable initial segments: if a countable suitable model contains $Y_{\alpha+m} \cap \beta$ and has $\omega_1^M = \beta$, the formula $\psi(\alpha, m, Y_{\alpha+m})$ gives the two corresponding ϑ-statements.

It remains to recall why diamond survives. In L, define a $\diamondsuit'$-sequence by letting, for $\alpha < \omega_1$, $\beta(\alpha)$ be the least β such that $L_\beta \models \mathsf{ZF}^- +$"$\alpha$ is countable", and set

$$D^0_\alpha = \{A \subseteq \alpha \mid A \in L_{\beta(\alpha)}\}.$$

Let $\dot{X}$ be a $\mathbb{P}^*$-name for a subset of ω_1 and let $\dot{C}$ be a name for a club. Choose the least countable elementary submodel N of a large L_Θ containing the relevant parameters. By the standard fusion argument used in the proof that $\mathbb{P}^*$ adds no reals, a condition can be strengthened so as to meet all dense sets in N. If $\alpha = N \cap \omega_1$, then the collapsed N and the induced generic over it are coded in $L_{\beta(\alpha)}$. The same strengthening therefore forces

$$\dot{X} \cap \alpha \in D^0_\alpha \quad \text{and} \quad \alpha \in \dot{C}.$$

Thus $\langle D^0_\alpha \mid \alpha < \omega_1\rangle$ is still a $\diamondsuit'$-sequence in L^*. Kunen's standard conversion of $\diamondsuit'$ to $\diamondsuit$ [Kun80, Theorem II.7.14] gives the last claim, and the construction is Σ_1 over L_{ω_1} because the map $\alpha \mapsto \beta(\alpha)$ is. □

2.2 Laver-style almost-disjoint coding

Fix in L^* a definable almost disjoint family

$$\vec{c} = \langle c_\xi \mid \xi < \omega_1\rangle$$

of infinite subsets of ω. If $A \subseteq \omega_1$, let I_A be the ideal on ω generated by the finite sets together with $\{c_\xi \mid \xi \in A\}$; equivalently,

$$b \in I_A \iff \exists F \in [A]^{<\omega}\ b \subseteq^* \bigcup_{\xi \in F} c_\xi.$$

Let F_A be the dual filter:

$$X \in F_A \iff \omega \setminus X \in I_A.$$

Definition 2.2. *For $A \subseteq \omega_1$, let $\mathbb{L}_A(\vec{c})$ be the set of all trees $T \subseteq \omega^{<\omega}$ such that T has a stem and, for every node $t \in T$ above the stem,*

$$\mathrm{Succ}_T(t) = \{n \mid t^\frown n \in T\} \in F_A.$$

The order is inclusion.

If $G \subseteq \mathbb{L}_A(\vec{c})$ is generic, let $x_G \in \omega^\omega$ be the union of the stems of conditions in G. The intended decoding is

$$\xi \in A \iff |\operatorname{ran}(x_G) \cap c_\xi| < \omega.$$

Lemma 2.3 (Fischer–Friedman–Khomskii)**.** *For every $A \subseteq \omega_1$, the forcing $\mathbb{L}_A(\vec{c})$ is σ-centered, adds a dominating real, strongly preserves splitting reals, and codes A by its generic branch.*

Proof. We prove the four assertions separately.

First, $\mathbb{L}_A(\vec{c})$ is σ-centered. For each finite sequence $s \in \omega^{<\omega}$, the set of conditions with stem s is centered: if $T_0, \ldots, T_{k-1}$ have stem s, then their intersection is again a condition with stem s, because F_A is a filter and hence is closed under finite intersections. Since there are only countably many possible stems, the forcing is σ-centered.

Secondly, the generic branch is dominating over the ground model. Let $f \in \omega^\omega$ and let T be a condition. We recursively prune T to a subtree $S \leq T$ so that, whenever t is a node of length n above the stem in S, every immediate successor m of t in S satisfies $m > f(n)$. This pruning is legitimate because deleting finitely many integers from an F_A-positive successor set leaves a member of F_A. Thus below S the generic branch is eventually above f. Since f was arbitrary, x_G dominates every ground-model real.

Thirdly, $\mathbb{L}_A(\vec{c})$ strongly preserves splitting reals. We use the Brendle–Hrušák criterion for Laver forcing with a filter [BH09, Proposition 1]. The criterion says that Laver forcing with the dual filter of an ideal I has the s.p.s. property provided that, for every $X \in I^+$ and every ideal J Katětov-below $I \restriction X$, the ideal J is not countably tall. For the ideal I_A this condition follows from almost disjointness. Indeed, let $X \in I_A^+$ and suppose $f : X \to \omega$ witnesses $J \leq_K I_A \restriction X$. If X has infinite intersection with infinitely many generators c_{ξ_n}, $\xi_n \in A$, put $a_n = f``(X \cap c_{\xi_n})$. For every $b \in J$ we have $f^{-1}[b] \in I_A \restriction X$, so $f^{-1}[b]$ is almost contained in a finite union of generators from A. Hence it has infinite intersection with only finitely many of the pairwise almost disjoint sets c_{ξ_n}, and therefore b has finite intersection with some a_n. This sequence witnesses that J is not countably tall. If X meets only finitely many generators from A in an infinite set, then after removing a set in I_A the restricted ideal is just the finite ideal on an infinite set; the same conclusion is immediate. Thus the criterion applies.

Finally we verify the coding. Let $\xi \in A$ and $T \in \mathbb{L}_A(\vec{c})$. Prune T above its stem by replacing each successor set S_t with $S_t \setminus c_\xi$. Since $c_\xi \in I_A$, the complement of $S_t \setminus c_\xi$ is still in I_A, so the pruned tree is a condition. It forces that all sufficiently late values of the generic branch avoid c_ξ; hence $|\operatorname{ran}(x_G) \cap c_\xi| < \omega$.

Conversely, suppose $\xi \notin A$. Let T be a condition and let $k < \omega$. For any node t above the stem, $\operatorname{Succ}_T(t) \in F_A$, and therefore $\omega \setminus \operatorname{Succ}_T(t) \in I_A$. Since c_ξ is almost disjoint from each c_η with $\eta \in A$, it is not an element of I_A. Hence $\operatorname{Succ}_T(t) \cap c_\xi$ is infinite. We may therefore extend the stem through some $m \in c_\xi$ with $m > k$. This proves that the generic branch hits c_ξ infinitely often. The decoding equivalence follows. □

2.3 The s.p.s. preservation theorem

We shall use the following abbreviation throughout the rest of the paper. A real $z \subseteq \omega$ *splits* an infinite set $a \subseteq \omega$ if both $z \cap a$ and $a \setminus z$ are infinite.

Definition 2.4. *A forcing $\mathbb{P}$ has the* s.p.s. property *if for every $\mathbb{P}$-name $\dot{a}$ for an infinite subset of ω there is a sequence $\langle a_n \mid n < \omega\rangle$ of infinite subsets of ω in the ground model such that every real splitting all a_n is forced to split $\dot{a}$.*

The reason this is the correct preservation notion is that it transfers splitting over a countable model through the forcing.

Lemma 2.5. *Let M be a countable transitive model, let $\mathbb{P} \in M$ be ccc and s.p.s. in M, and let $z \subseteq \omega$ split every infinite subset of ω belonging to M. If G is $\mathbb{P}$-generic over M, then z splits every infinite subset of ω belonging to $M[G]$.*

Proof. Let $a \in M[G]$ be infinite and choose in M a $\mathbb{P}$-name $\dot{a}$ for it. By s.p.s. in M, there is a sequence $\langle a_n \mid n < \omega\rangle \in M$ of infinite sets such that every real splitting all the a_n is forced to split $\dot{a}$. The real z splits all the a_n, since they belong to M. Hence $M[G] \models$ "z splits a". □

Lemma 2.6 (Brendle)**.** *The s.p.s. property is preserved by finite-support iterations of ccc forcings [Bre09, Proposition 3.10]. In particular every finite-support iteration whose iterands are Cohen forcing, trivial forcing, or forcings of the form $\mathbb{L}_A(\vec{c})$ has the s.p.s. property.*

Proof. For a two-step iteration the argument is transparent. Suppose $\mathbb{P}$ is ccc and s.p.s., and

$$\mathbb{P} \Vdash \text{“}\dot{\mathbb{Q}} \text{ is ccc and s.p.s.''}.$$

Let $\dot{a}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for an infinite subset of ω. In the $\mathbb{P}$-extension, apply s.p.s. for $\dot{\mathbb{Q}}$ to obtain a sequence of $\mathbb{P}$-names $\langle \dot{b}_n \mid n < \omega\rangle$ such that every real splitting all $\dot{b}_n$ is forced by $\dot{\mathbb{Q}}$ to split $\dot{a}$. Now apply s.p.s. for $\mathbb{P}$, in the ground model, to each name $\dot{b}_n$. This yields a double sequence $\langle b_{n,k} \mid n, k < \omega\rangle$ of ground-model infinite sets such that every real splitting all $b_{n,k}$ is forced by $\mathbb{P}$ to split every $\dot{b}_n$. Therefore every real splitting the single enumerated sequence $\langle b_{n,k} \mid n, k < \omega \rangle$ is forced by the two-step iteration to split $\dot{a}$.

Successor stages follow by repeating this argument. At a limit stage of a finite-support ccc iteration, every name for a real is supported on a countable set of coordinates. Replacing this countable support by an increasing sequence of successor stages and applying the preceding paragraph recursively gives a countable family of ground-model sets which witnesses s.p.s. for the given name. This proves the preservation theorem. Since Cohen forcing and trivial forcing are s.p.s., and since Lemma 2.3 gives s.p.s. for every $\mathbb{L}_A(\vec{c})$, the final assertion follows. □

2.4 The perfect mad family

We now recall the Brendle–Khomskii construction in the form needed later. If $D \in [\omega]^\omega$ and

$$P = \{P_s \mid s \in \omega^{<\omega}\}$$

is a partition of D into infinite sets indexed by finite sequences, write $p_s(i)$ for the increasing enumeration of P_s and define

$$\Phi_P(f) = \{p_{f\restriction n}(f(n)) \mid n < \omega\}, \qquad f \in \omega^\omega.$$

The range

$$\mathcal{A}_P = \{\Phi_P(f) \mid f \in \omega^\omega\}$$

is a perfect almost disjoint family on D. In the construction we write these as P^α, Φ_α and $\mathcal{A}_\alpha$.

Lemma 2.7 (Brendle–Khomskii main lemma)**.** *Let M be a countable transitive model containing partitions P^β for all $\beta < \alpha$, and suppose that $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ is almost disjoint. Then there is a partition $P^\alpha \notin M$ such that:*

1. *$\bigcup_{\beta\leq\alpha} \mathcal{A}_\beta$ is almost disjoint;*
2. *if $Y \in M \cap [\omega]^\omega$ is almost disjoint from every member of $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$, then for some $h \in \omega^\omega$ one has $\Phi_\alpha(h) \subseteq Y$;*
3. *the preceding clause is absolute to extensions $M \subseteq M'$ in which every real splitting over M still splits over M'. More precisely, if $V \subseteq V'$ and $M \subseteq M' \in V'$ are as in the Brendle–Khomskii lemma, and every real of V splitting over M remains splitting over M', then for every $Y \in M' \cap [\omega]^\omega$ almost disjoint in V' from all earlier perfect pieces, there is $h \in (\omega^\omega)^{V'}$ such that $\Phi_\alpha(h) \subseteq Y$.*

Proof. This is the main combinatorial lemma of Brendle–Khomskii [BK13, Lemma 3.4]. The construction of P^α is a fusion construction over the countable model M. One enumerates the relevant sets $Y \in M$ which are almost disjoint from the earlier perfect pieces and, at successive stages, reserves infinitely many fresh points inside the current Y while keeping the new partition almost disjoint from all old branches $\Phi_\beta(f)$. The splitting hypothesis is used to guarantee that the choices made outside M can be arranged so as to avoid all old perfect pieces uniformly. The same fusion works in the relativized situation because the assumption that splitting reals over M remain splitting over M' is exactly what is needed to repeat the avoidance argument for the new sets in M'. □

An $\aleph_1$-perfect mad family is a family of the form

$$\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha,$$

where each $\mathcal{A}_\alpha$ is perfect and the union is mad. The next result is the preservation theorem that will be invoked in the final verification.

Theorem 2.8. *Let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \delta \rangle, \qquad |\delta| \leq \aleph_3,$$

be a finite-support iteration in L^* such that, for every $\alpha < \delta$,

$$\mathbb{P}_\alpha \Vdash \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is } \sigma\textit{-centered and has the s.p.s. property.”}$$

Then there is, in L^*, an $\aleph_1$-perfect Σ^1_2 mad family

$$\mathcal{A} = \bigcup_{\xi<\omega_1} \mathcal{A}_\xi$$

such that

$$\mathbb{P}_\delta \Vdash \text{“}\mathcal{A}\textit{, reinterpreted as } \bigcup_{\xi<\omega_1} \mathcal{A}_\xi^{V[\dot{G}]}\textit{, is mad.”}$$

Moreover the same Σ^1_2 definition reinterprets the family in every $\mathbb{P}_\delta$-generic extension.

Proof. This is the form of the Fischer–Friedman–Khomskii preservation argument needed below. The same proof applies to any finite-support iteration whose index set has cardinality at most $\aleph_3$, in particular to the combined prelude-plus-main forcing used in the final section. We recall the proof because the point is exactly where the s.p.s. hypothesis is used.

First, by Lemma 2.6, the full iteration $\mathbb{P}_\delta$ has the s.p.s. property. Fix in L^* the Σ_1-definable $\Diamond'$ sequence supplied by the preliminary preparation from [FFK13]. The construction of the family is carried out by induction on $\alpha < \omega_1$. At stage α, the diamond guess is read, if possible, as a code for a countable transitive model M_α containing the previous partitions, a sufficiently large finite fragment of the iteration, and the names for infinite subsets of ω which are predicted at that stage. If the guess is not suitable, we take the $<_{L^*}$-least countable transitive model with the required closure. Applying the Brendle–Khomskii main lemma (Lemma 2.7) to M_α and the previously constructed partitions produces a new partition P^α and hence a perfect almost-disjoint family $\mathcal{A}_\alpha$.

The induction preserves almost disjointness by Lemma 2.7(1). The usual diamond argument gives maximality in L^*: if $Y \in L^* \cap [\omega]^\omega$ is almost disjoint from all earlier pieces, then at a stationary set of stages the diamond sequence guesses a countable model containing Y and the relevant initial segment of the construction. At such a stage, Lemma 2.7(2) yields $h \in \omega^\omega$ with $\Phi_\alpha(h) \subseteq Y$, contradicting that Y is almost disjoint from the final union.

Now let $G \subseteq \mathbb{P}_\delta$ be generic over L^*, and let $Y \in L^*[G] \cap [\omega]^\omega$. Choose a nice $\mathbb{P}_\delta$-name $\dot{Y}$ for Y. Since the iteration is c.c.c. and finite-support, $\dot{Y}$ is supported by a countable set of coordinates. The diamond sequence guesses stationarily often a countable model M_α containing this support, the name $\dot{Y}$, and the relevant initial segment of the iteration. Let G_α^M be the generic induced over this countable model. By Lemma 2.5, every real which split all infinite subsets of ω in M_α before forcing still splits every infinite subset of ω in $M_\alpha[G_\alpha^M]$. Thus the relativized clause of Lemma 2.7 applies in the extension.

If Y were almost disjoint from the reinterpretation of every earlier perfect piece, Lemma 2.7(3), applied to $M_\alpha[G_\alpha^M]$, would produce some $h \in (\omega^\omega)^{L^*[G]}$ such that

$$\Phi_\alpha(h) \subseteq Y.$$

But $\Phi_\alpha(h)$ belongs to the reinterpreted perfect family $\mathcal{A}_\alpha$, hence is a member of the reinterpreted union $\mathcal{A}$. This contradicts the assumption that Y is almost disjoint from $\mathcal{A}$. Therefore the reinterpretation of $\mathcal{A}$ remains maximal in $L^*[G]$.

Finally, the construction is Σ^1_2-definable. Membership in $\mathcal{A}$ says that there is a countable collapse which verifies the corresponding stage of the diamond construction and that the real belongs to the perfect set produced from the partition at that stage. Once the collapse and the branch through the partition tree are given, the verification is arithmetical. The preserved Σ_1 definition of the diamond sequence over L_{ω_1} therefore gives a Σ^1_2 definition, and ccc forcing reinterprets the same perfect trees without changing the defining formula. □

By Törnquist's theorem [Tör13], the existence of a Σ^1_2 mad family is equivalent to the existence of a Π^1_1 mad family. Thus, once the final iteration is shown to be ccc and s.p.s., the final model contains a co-analytic mad family.

3 The localized Σ^1_3 coding predicate

We isolate only the interface of the localized coding construction from [FZ10, FFZ11, FFK13] which is used below. The preparatory extension L^* and the localized sets $Y_\xi \subseteq \omega_1$, $\xi < \omega_3$, are those of Section 2. The sequence $\vec{c} = \langle c_\xi \mid \xi < \omega_1 \rangle$ is the fixed definable almost disjoint family used for almost-disjoint coding.

Fix once and for all a recursive coding of arbitrary reals by subsets of ω; write $t_x \subseteq \omega$ for the code of the real x. If $x \subseteq \omega$ already, we may take $t_x = x$. For $t \subseteq \omega$, put

$$\Delta(t) = \{2n+2 \mid n \in t\} \cup \{2n+1 \mid n \notin t\},$$

and write $\Delta(x)$ for $\Delta(t_x)$. Thus $\Delta(t)$ chooses exactly one member of each pair $\{2n+1, 2n+2\}$, and never chooses 0.

Lemma 3.1. *If $s, t \subseteq \omega$ and $\Delta(s) \subseteq \Delta(t)$, then $s = t$.*

Proof. For each n, both sets contain exactly one element of $\{2n+1, 2n+2\}$. Inclusion therefore forces equality on every pair. □

Definition 3.2 (The block forcing)**.** *Let x be a real in the current intermediate model, and let $\eta \in \mathrm{Lim}(\omega_3)$ be a fresh block. The forcing*

$$\mathrm{Code}(\eta, x)$$

is the finite-support iteration $\langle \mathbb{Q}^m_{\eta,x} \mid m < \omega \rangle$, where

$$\mathbb{Q}^m_{\eta,x} = \begin{cases} \mathbb{L}_{Y_{\eta+m}}(\vec{c}), & m \in \Delta(x), \\ \textit{the trivial forcing}, & m \notin \Delta(x). \end{cases}$$

Freshness means that no other coding stage uses any coordinate $\eta+m$, $m < \omega$.

Lemma 3.3. *For every real x and every fresh block η, the forcing $\mathrm{Code}(\eta, x)$ is σ-centered and has the s.p.s. property. It also adds a dominating real.*

Proof. Each non-trivial factor is one of the Laver-like almost-disjoint coding forcings $\mathbb{L}_A(\vec{c})$ from Lemma 2.3, with $A = Y_{\eta+m}$. Hence each such factor is σ-centered, s.p.s., and adds a dominating real. Finite-support iterations of length ω preserve σ-centeredness and s.p.s. by Lemma 2.6. Since $\Delta(x) \neq \varnothing$, at least one non-trivial factor is present. □

Definition 3.4 (The predicate Coded)**.** $\mathrm{Coded}(x)$ *is the localized coding predicate associated with the blocks $Y_{\eta+m}$, as in [FZ10, FFZ11, FFK13]. Equivalently, $\mathrm{Coded}(x)$ asserts that there is a real R coding a sequence $\langle u_m \mid m < \omega \rangle$ such that the usual countable suitable-model test verifies one common localized block η and verifies, for every $m \in \Delta(x)$, that u_m almost-disjointly* *codes the localized coordinate $Y_{\eta+m}$ of that block. This is the suitable-model formula used to obtain the Σ^1_3 half of the Δ^1_3 well-order, with the Laver-like coding of Lemma 2.3 in place of the standard almost-disjoint coding.*

We shall often keep the older notation

$$\Phi(x) \iff \mathrm{Coded}(x),$$

and write

$$\mathsf{C}(x) \quad \textit{for} \quad \mathrm{Coded}(x), \qquad \mathsf{N}(x) \quad \textit{for} \quad \neg\,\mathrm{Coded}(x).$$

These are not new predicates.

Lemma 3.5. *The predicate* $\mathrm{Coded}(x)$*, equivalently* $\Phi(x)$*, is* Σ^1_3*. Its negation is* Π^1_3*.*

Proof. This is the complexity calculation for the localized block predicate in [FFZ11, FFK13]. The witness is one real coding the relevant sequence of almost-disjoint coding reals; the remaining verification is the standard universal quantification over countable suitable models followed by a first-order check inside the collapse. The same calculation applies after replacing standard almost-disjoint coding by the Laver-like coding of Lemma 2.3. □

Lemma 3.6. *Let* $W \subseteq W'$ *be transitive models of a sufficiently large fragment of set theory with the same* ω_1*. If* $W \models \mathrm{Coded}(x)$*, then* $W' \models \mathrm{Coded}(x)$*.*

Proof. For a fixed real witness R, the localized block verification is Π^1_2 in (R, x). Shoenfield absoluteness therefore preserves the witness from W to W'. □

Lemma 3.7. *If an intermediate stage forces with* $\mathrm{Code}(\eta, x)$*, where* η *is fresh, then* $\mathrm{Coded}(x)$ *holds in the resulting extension and remains true in all later c.c.c. extensions preserving* ω_1*.*

Proof. The generic reals added at the coordinates $\eta + m$, $m \in \Delta(x)$, supply the real witness in Definition 3.4. Persistence is Lemma 3.6. □

Lemma 3.8 (No unwanted codes)**.** *Let*

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \theta \rangle$$

be a finite-support iteration over L^* *such that every non-trivial iterand is of the form*

$$\dot{\mathbb{Q}}_\alpha = \mathrm{Code}(\eta_\alpha, \dot{x}_\alpha),$$

where the blocks η_α *are fresh and pairwise disjoint, and all other iterands are trivial. If* $G \subseteq \mathbb{P}$ *is generic, then in* $L^*[G]$*,*

$$\mathrm{Coded}(x) \iff \exists \alpha < \theta \left(\dot{\mathbb{Q}}_\alpha = \mathrm{Code}(\eta_\alpha, \dot{x}_\alpha) \textit{ and } x = \dot{x}^G_\alpha\right).$$

The same equivalence holds in every complete subextension generated by any set of coding coordinates, with the existential quantifier restricted to the stages present in that subiteration.

Proof. This is the no-unwanted-codes theorem for the localized coding construction, in the Laver-like form used for the Δ^1_3 well-order in [FFZ11, FFK13]. In that version standard almost-disjoint coding is replaced by the Laver-like forcing of Lemma 2.3, and the verification of the coding stage, including the relevant no-unwanted-code analysis, goes through unchanged. The proof is local to the coordinates used by the real name, so it also gives the stated complete-subiteration form. □

4 The combined forcing

We now define the final forcing. The base predicate is Coded, also denoted Φ, and the block forcing is $\mathrm{Code}(\eta, x)$, as in Section 3. The purpose of this section is only to fix the bookkeeping and the stage rules.

Definition 4.1 (Tags, reservoirs and stage classes)**.** *The reals intentionally fed into* Coded *are tagged reals, using a fixed recursive coding of finite tuples. The allowed forms are*

$$\begin{aligned}
&\text{well-order tags:} && \langle \mathrm{wo}, x, y\rangle,\\
&\text{dummy tags:} && \langle \mathrm{dummy}, 0\rangle,\\
&\text{uniformization tags:} && \langle \mathrm{unif}, \varepsilon, m, x, y, a_0, b_1, \dots, b_k\rangle,\\
& && \varepsilon \in \{\mathrm{odd}, \mathrm{even}\}, \quad m, k < \omega.
\end{aligned}$$

For such a tag z, $t_z \subseteq \omega$ denotes its recursive code. We write

$$\mathrm{Tag}(\langle \mathrm{wo}, x, y\rangle) = \mathrm{wo}, \qquad \mathrm{Tag}(\langle \mathrm{dummy}, 0\rangle) = \mathrm{dummy},$$

and

$$\mathrm{Tag}(\langle \mathrm{unif}, \varepsilon, m, x, y, a_0, b_1, \dots, b_k\rangle) = (\mathrm{unif}, \varepsilon, m).$$

Fix, once and for all, a partition of ω_3 into pairwise disjoint cofinal sets

$$E_{\mathrm{wo}}, \qquad E_{\mathrm{unif}}, \qquad E_{\mathrm{dom}}$$

and fix pairwise disjoint cofinal reservoirs

$$\mathcal{R}_{\mathrm{wo}}, \qquad \mathcal{R}_{\mathrm{unif}}, \qquad \mathcal{R}_{\mathrm{pre}}, \qquad \mathcal{R}_{\mathrm{dom}}, \qquad \mathcal{R}_{\mathrm{extra}}$$

inside $\mathrm{Lim}(\omega_3)$*. Put*

$$\mathcal{R}_{\mathrm{dummy}} = \mathcal{R}_{\mathrm{pre}} \sqcup \mathcal{R}_{\mathrm{dom}} \sqcup \mathcal{R}_{\mathrm{extra}},$$

and define

$$\begin{array}{rcl}\mathrm{Res}(\mathrm{wo}) & = & \mathcal{R}_{\mathrm{wo}},\\ \mathrm{Res}(\mathrm{dummy}) & = & \mathcal{R}_{\mathrm{dummy}},\\ \mathrm{Res}(\mathrm{unif},\varepsilon,m) & = & \mathcal{R}_{\mathrm{unif}} \qquad (\varepsilon\in\{\mathrm{odd},\mathrm{even}\},\ m<\omega).\end{array}$$

We also fix injections

$$\alpha\mapsto\rho^{\mathrm{pre}}_\alpha \quad (\alpha<\omega_3), \qquad \alpha\mapsto\rho_\alpha \quad (\alpha\in E_{\mathrm{wo}}\cup E_{\mathrm{unif}}\cup E_{\mathrm{dom}}),$$

such that

$$\rho^{\mathrm{pre}}_\alpha\in\mathcal{R}_{\mathrm{pre}}, \qquad \rho_\alpha\in\mathcal{R}_{\mathrm{wo}}\ (\alpha\in E_{\mathrm{wo}}), \qquad \rho_\alpha\in\mathcal{R}_{\mathrm{unif}}\ (\alpha\in E_{\mathrm{unif}}), \qquad \rho_\alpha\in\mathcal{R}_{\mathrm{dom}}\ (\alpha\in E_{\mathrm{dom}}).$$

Thus every prelude or main stage has a preassigned fresh block from the correct reservoir; if the main stage is trivial, that block is simply left unused.

We first run a dummy prelude over L^*:

$$\mathbb{P}^{\mathrm{pre}}=\langle\mathbb{P}^{\mathrm{pre}}_\alpha,\dot{\mathbb{Q}}^{\mathrm{pre}}_\alpha\mid\alpha<\omega_3\rangle, \qquad \dot{\mathbb{Q}}^{\mathrm{pre}}_\alpha=\mathrm{Code}(\rho^{\mathrm{pre}}_\alpha,\langle\mathrm{dummy},0\rangle),$$

with finite support. Its only role is to make the compression convention below available at every later main stage. Let

$$L^\dagger=L^*[G^{\mathrm{pre}}].$$

The main recursion is carried out over $L^\dagger$. For statements about exactness of the coding we regard the whole forcing over L^* as

$$\mathbb{P}^{\mathrm{tot}}=\mathbb{P}^{\mathrm{pre}}*\dot{\mathbb{P}}^{\mathrm{main}}_{\omega_3},$$

or, equivalently, as the corresponding flattened finite-support iteration. The prelude uses only dummy tags and blocks from $\mathcal{R}_{\mathrm{pre}}$, so it cannot create well-order or uniformization tags.

Lemma 4.2. $\mathbb{P}^{\mathrm{pre}}$ *is a finite-support iteration of* σ*-centered s.p.s. forcings. In* $L^\dagger$*,*

$$2^\omega=2^{\omega_1}=2^{\omega_2}=\aleph_3, \qquad (\aleph_3)^{\aleph_i}=\aleph_3 \quad (i=0,1,2).$$

Moreover the co-analytic mad family prepared in Section 2 remains maximal after the prelude.

Proof. Each factor is a block forcing from Lemma 3.3; hence it is σ-centered, c.c.c., adds a dominating real, and is s.p.s. Finite-support iterations preserve c.c.c. and the s.p.s. property. The standard simultaneous size and nice-name induction gives $|\mathbb{P}^{\mathrm{pre}}|\le\aleph_3$: if the current continuum is at most $\aleph_3$, then the next block forcing has size at most $\aleph_3$, and the successor and limit steps are the usual finite-support ones.

Since the forcing is c.c.c. and has size at most $\aleph_3$, nice-name counting over L^*, together with Lemma 2.1, gives $2^{\omega_i}\le\aleph_3$ for $i=0,1,2$. The prelude has $\aleph_3$ many fresh non-trivial blocks, hence adds $\aleph_3$ many distinct reals; this gives the reverse inequalities. The displayed power identities follow immediately. The last assertion is the s.p.s. preservation theorem from Section 2. □

For the main iteration, if $G_\alpha \subseteq \mathbb{P}_\alpha$ is generic over $L^\dagger$, set

$$W_\alpha = L^\dagger[G_\alpha].$$

All names at stage α are $\mathbb{P}_\alpha$-names over $L^\dagger$. Since the main iteration has finite support and c.c.c. iterands, every real in the final extension appears in some W_α, $\alpha < \omega_3$.

4.1 Bookkeeping, blocks and intermediate orders

Stages in E_{wo} create the well-order, stages in E_{unif} run the uniformization bookkeeping, and stages in E_{dom} add dummy dominating reals. The corresponding blocks ρ_α were fixed in Definition 4.1; no block is reused.

We use three standard bookkeepings.

(i) B_{wo} lists pairs of names for reals.

(ii) B_{unif} lists finite tuples

$$(m, \varepsilon, \dot{x}, \dot{y}, \dot{a}_0, \dot{b}_1, \ldots, \dot{b}_k),$$

where m is a formula code and $\varepsilon \in \{0, 1\}$ records the parity. The reals b_i are compression variables; their decoded sequences are obtained in the current intermediate model from the maps π^α_ξ fixed in Subsection 4.2.

(iii) B_{dom} is the constant dummy bookkeeping.

The bookkeepings have the usual capture property: every finite tuple of real names, and every formula code needed in the construction, is listed cofinally often after the supports of the names have become bounded. The bounded-support fact used here is recorded below.

For $r \in W_\alpha$, let

$$\mathrm{rk}_\alpha(r) = \min\{\beta \leq \alpha : r \in W_\beta\}.$$

Define $<^\alpha_{\mathrm{can}}$ on the reals of W_α by first comparing rk_α, and then, at equal rank β, by comparing the $<_L$-least $\mathbb{P}_\beta$-names for the two reals in the fixed ground-model order of names.

Lemma 4.3. *If $\alpha < \beta$ and $r, s \in W_\alpha$, then*

$$r <^\alpha_{\mathrm{can}} s \iff r <^\beta_{\mathrm{can}} s.$$

Proof. Later forcing cannot give an old real an earlier name in the same finite-support iteration, and the ground-model tie-breaker on names is fixed. Thus both the first stage at which the real appears and the tie-breaker are unchanged. □

For uniformization we slightly modify the induced order on triples with a fixed first coordinate. For each real x, the first triple is forced to be

$$t_0^x = (x, 0, 0),$$

where 0 is the recursive zero real. All other triples $(x, y, a_0) \neq (x, 0, 0)$ are ordered by the canonical order of the pair (y, a_0). We denote the resulting order by $<_{\mathrm{tr}}^{\alpha,x}$. Its coherence in α follows from Lemma 4.3. This convention ensures that, whenever 0 with witness $a_0 = 0$ already works over x, the eventual uniformizing value is forced to be 0.

4.2 Cardinal arithmetic and coherent compression

For $\beta \leq \omega_3$, put

$$R_\beta = 2^\omega \cap W_\beta.$$

Lemma 4.4. *For every $\beta \leq \omega_3$,*

$$W_\beta \models 2^\omega = 2^{\omega_1} = 2^{\omega_2} = \aleph_3.$$

Consequently, for every $0 < \xi < \omega_3$,

$$W_\beta \models |R_\beta^\xi| = |R_\beta|.$$

Proof. By the same simultaneous size and nice-name induction as for the prelude, every $\mathbb{P}_\beta$ is c.c.c. and has size at most $\aleph_3$: non-trivial iterands are block forcings, and the successor and limit size bounds are the standard finite-support ones. Nice-name counting over $L^\dagger$ then gives $2^{\omega_i} \leq \aleph_3$ in W_β for $i = 0, 1, 2$, using Lemma 4.2. The reverse inequalities are already true in $L^\dagger$. Finally $|\xi| \leq \aleph_2$, so

$$|R_\beta^\xi| = (\aleph_3)^{|\xi|} = \aleph_3 = |R_\beta|.$$

□

At each stage β, and for every $0 < \xi < \omega_3$, we use a bijection

$$\pi_\xi^\beta : R_\beta^\xi \longrightarrow R_\beta$$

which extends all earlier such bijections on old sequences. Thus, if $\beta < \gamma$ and $\vec{a} \in R_\beta^\xi$, then

$$\pi_\xi^\gamma(\vec{a}) = \pi_\xi^\beta(\vec{a}),$$

and old compression reals decode to the same old sequences in later models. For $\xi = 0$ the decoded sequence is, by convention, empty.

Lemma 4.5 (Coherent compression maps)**.** *There is a coherent system*

$$\langle \pi^\beta_\xi \mid \beta \le \omega_3,\ 0 < \xi < \omega_3 \rangle$$

of bijections $\pi^\beta_\xi : R^\xi_\beta \to R_\beta$ *as above.*

Proof. Construct the maps by recursion on β. At stage β, the union of the previous maps is an injection from the old ξ-sequences of reals into the old reals. By Lemma 4.4, both R^ξ_β and R_β have size $\aleph_3$, so this injection extends to a bijection. Choose one such extension for each ξ by the fixed construction well-order. □

We write $(\pi^\beta_\xi)^{-1}(b)$ for the decoded ξ-sequence of reals. The maps are auxiliary construction data only; the final projective formulas mention only the finite tags containing the compression reals.

Lemma 4.6. *Every* $\mathbb{P}_{\omega_3}$*-name for a sequence of reals of length* $\xi < \omega_3$ *is equivalent to a* $\mathbb{P}_\beta$*-name for some* $\beta < \omega_3$*.*

Proof. Code the sequence as a name for a subset of $\xi \times \omega$. Choose countable maximal antichains deciding each statement $n \in \dot r_\eta$. The union of the finite supports appearing in these antichains has size at most $|\xi| \cdot \aleph_0 \le \aleph_2$, hence is bounded in the regular cardinal ω_3. Restricting to a sufficiently large initial segment gives an equivalent name. □

We shall use Shoenfield absoluteness: if $\alpha < \gamma \le \omega_3$, all real parameters lie in W_α, and χ is Σ^1_2 or Π^1_2, then

$$W_\alpha \models \chi(\vec r) \quad \Longleftrightarrow \quad W_\gamma \models \chi(\vec r).$$

Together with coherent decoding, this makes the matrix tests used at odd and even uniformization stages stable in all later intermediate models and in the final model.

4.3 Recursive definition of the iteration

We define by recursion a finite-support iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \le \omega_3 \rangle.$$

At every non-trivial stage the iterand is of the form $\mathrm{Code}(\rho_\alpha, \dot z)$ for a tagged real $\dot z$ in the current intermediate model. Its interpreted tag determines the allowed reservoir in the sense of Definition 4.1, and the block ρ_α has already been chosen from that reservoir.

Well-order stages. Suppose $\alpha \in E_{\text{wo}}$ and $B_{\text{wo}}(\alpha) = (\dot{x}, \dot{y})$. If $\dot{x}$ and $\dot{y}$ are not both $\mathbb{P}_\alpha$-names for reals, the stage is trivial. Otherwise, in W_α let $x = \dot{x}^{G_\alpha}$ and $y = \dot{y}^{G_\alpha}$. If $x = y$, the stage is trivial. If $x <^\alpha_{\text{can}} y$, set

$$z_\alpha = \langle \text{wo}, x, y \rangle$$

and force with $\text{Code}(\rho_\alpha, z_\alpha)$. If $y <^\alpha_{\text{can}} x$, set

$$z_\alpha = \langle \text{wo}, y, x \rangle$$

and force with $\text{Code}(\rho_\alpha, z_\alpha)$. This is a legitimate definition of a $\mathbb{P}_\alpha$-name for the next iterand by the forcing theorem, since $<^\alpha_{\text{can}}$ is definable in W_α.

The final well-order is defined from the base predicate by

Definition 4.7. *In the final extension set*

$$x <_\Delta y \iff \Phi(\langle \text{wo}, x, y \rangle).$$

The coherence lemma above is the reason repeated appearances of the same pair in the bookkeeping do not create incompatible information. Once both reals are present, every later well-order stage computes the same orientation.

Dominating stages. If $\alpha \in E_{\text{dom}}$, we force with

$$\text{Code}(\rho_\alpha, \langle \text{dummy}, 0 \rangle).$$

The same dummy tag may be used at many different blocks. This is harmless: the only purpose of these stages is to add dominating reals cofinally often. Since the block is fresh, the forcing still adds a new dominating real even if the dummy tag has already been base-coded at an earlier stage.

Uniformization stages. We now describe the uniformization stages in detail. The rule follows the "copied infinitary test" pattern from [Hof25a].

Let m code a projective formula in the plane and fix an intermediate model W_α. For a triple $t = (x, y, a_0) \in W_\alpha$, let

$$I^\alpha_t = \{t' = (x, y', a'_0) : t' <^{\alpha,x}_{\text{tr}} t\}$$

be the set of earlier triples with the same first coordinate, ordered with the zeroth-triple convention. Put

$$\xi^\alpha_t = \text{otp}(I^\alpha_t).$$

If $\xi^\alpha_t > 0$ and $b \in W_\alpha$, then $(\pi^\alpha_{\xi^\alpha_t})^{-1}(b)$ is a well-defined sequence indexed by I^α_t. We write it as

$$(\pi^\alpha_{\xi^\alpha_t})^{-1}(b) = \langle a^{t'}_b : t' \in I^\alpha_t \rangle.$$

If $\xi_t^\alpha = 0$, this displayed decoding is replaced by the empty sequence. Thus the zeroth triple $(x, 0, 0)$ has a vacuous copied-failure test for every choice of compression reals.

Odd levels. Suppose

$$\varphi_m(x, y) \equiv \exists a_0\, \forall a_1\, \exists a_2 \cdots \exists a_{2n-2}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-2}),$$

where $n \geq 1$ and $\psi_m \in \Pi^1_2$. For $t = (x, y, a_0)$ put

$$\Theta_m(t) \quad \Longleftrightarrow \quad \forall a_1 \exists a_2 \cdots \exists a_{2n-2}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-2}).$$

Thus $(x, y) \in A_m$ iff $\Theta_m(x, y, a_0)$ holds for some a_0. Set

$$\ell_n^O = \max(1, 2n-2).$$

A tuple $\vec{b} = (b_1, \ldots, b_{\ell_n^O})$ is said to pass the *odd copied failure test* for t in W_α if the following holds. If $n = 1$, then

$$W_\alpha \models \forall t' = (x, y', a_0') \in I_t^\alpha\ \neg\psi_m(x, y', a_0'). \tag{O$_1$}$$

If $n > 1$, then for every $1 \leq i \leq 2n-2$ decode

$$(\pi^\alpha_{\xi_t^\alpha})^{-1}(b_i) = \langle a_i^{t'} : t' \in I_t^\alpha \rangle$$

when $\xi_t^\alpha > 0$, and require

$$W_\alpha \models \forall t' = (x, y', a_0') \in I_t^\alpha\ \neg\psi_m(x, y', a_0', a_1^{t'}, \ldots, a_{2n-2}^{t'}). \tag{O}$$

When $I_t^\alpha = \emptyset$, both (O_1) and (O) are vacuous. The extra compression real in the case $n = 1$ is an auxiliary tag variable; no long sequence is decoded from it.

At a stage $\alpha \in E_{\mathrm{unif}}$, suppose

$$B_{\mathrm{unif}}(\alpha) = (m, 0, \dot{x}, \dot{y}, \dot{a}_0, \dot{b}_1, \ldots, \dot{b}_{\ell_n^O})$$

and suppose all displayed entries are $\mathbb{P}_\alpha$-names for reals of the appropriate sort. Let $x, y, a_0, b_1, \ldots, b_{\ell_n^O}$ be their interpretations in W_α, and put $t = (x, y, a_0)$. If the tuple $\vec{b} = (b_1, \ldots, b_{\ell_n^O})$ passes the odd copied failure test for t in W_α, set

$$z_\alpha = \langle \mathrm{unif}, \mathrm{odd}, m, x, y, a_0, b_1, \ldots, b_{\ell_n^O} \rangle$$

and force with $\mathrm{Code}(\rho_\alpha, z_\alpha)$. If the test fails, the stage is trivial. Notice that the stage rule does not need to decide the full success statement $\Theta_m(t)$; the final uniformizing formula includes that statement explicitly. The stage only copies the bounded family of earlier matrix failures.

Even levels. Suppose

$$\varphi_m(x, y) \equiv \exists a_0\, \forall a_1\, \exists a_2 \cdots \forall a_{2n-3}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-3}),$$

where $n \geq 2$ and $\psi_m \in \Sigma^1_2$. For $t = (x, y, a_0)$ put

$$\Theta_m(t) \iff \forall a_1 \exists a_2 \cdots \forall a_{2n-3}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-3}).$$

A tuple $\vec{b} = (b_1, \ldots, b_{2n-3})$ passes the *even copied failure test* for t in W_α if, for every $1 \leq i \leq 2n-3$,

$$(\pi^\alpha_{\xi^\alpha_t})^{-1}(b_i) = \langle a^{t'}_i : t' \in I^\alpha_t \rangle$$

when $\xi^\alpha_t > 0$, and the decoded sequences witness

$$W_\alpha \models \forall t' = (x, y', a'_0) \in I^\alpha_t\ \neg\psi_m(x, y', a'_0, a^{t'}_1, \ldots, a^{t'}_{2n-3}). \tag{E}$$

Again the test is vacuous when $I^\alpha_t = \emptyset$.

At an even uniformization stage presenting

$$(m, 1, \dot{x}, \dot{y}, \dot{a}_0, \dot{b}_1, \ldots, \dot{b}_{2n-3}),$$

let $x, y, a_0, b_1, \ldots, b_{2n-3}$ be the interpreted reals, let $t = (x, y, a_0)$, and set

$$z_\alpha = \langle \mathrm{unif}, \mathrm{even}, m, x, y, a_0, b_1, \ldots, b_{2n-3} \rangle.$$

If the even copied failure test (E) holds, the stage is trivial, so this particular tag is deliberately left uncoded. If (E) fails, the stage forces with $\mathrm{Code}(\rho_\alpha, z_\alpha)$ and hence deliberately codes the tag. This is the standard parity reversal: at odd levels the correct copied failure test is marked by coding, whereas at even levels it is marked by non-coding. Lemma 3.8 later turns these stage decisions into the final truth values of Φ and $\neg\Phi$.

The truth values of the displayed matrix statements in (O_1), (O), and (E) are preserved to later stages by Shoenfield absoluteness; coherence of the maps π^β_ξ ensures that an old compression real decodes to the same old sequence later.

This completes the recursive definition of $\dot{\mathbb{Q}}_\alpha$ and hence of $\mathbb{P}_{\omega_3}$.

4.4 Basic properties of the iteration

Lemma 4.8. *For every $\alpha \leq \omega_3$, $\mathbb{P}_\alpha$ is ccc. Moreover every non-trivial iterand is σ-centered and has the s.p.s. property, and the full iteration $\mathbb{P}_{\omega_3}$ has the s.p.s. property.*

Proof. At a non-trivial stage the iterand is $\mathrm{Code}(\rho_\alpha, z)$ for some tagged real z. By Lemma 3.3, this forcing is σ-centered and has the s.p.s. property. Finite-support iterations of ccc forcings are ccc, and finite-support iterations of ccc s.p.s. forcings are s.p.s. by Lemma 2.6. The assertion follows by induction on α. □

Lemma 4.9. *If r is a real in $L^\dagger[G]$, then $r \in W_\alpha$ for some $\alpha < \omega_3$. Consequently every finite tuple of reals in the final model is handled cofinally often by the relevant bookkeeping.*

Proof. Because $\mathbb{P}_{\omega_3}$ is a finite-support ccc iteration, every name for a real has countable support. The support is bounded in the regular cardinal ω_3. Thus the interpretation of the name belongs to the intermediate model determined by some bounded initial segment. The second assertion is exactly the capture property of the bookkeepings. □

Lemma 4.10. *Let $x \neq y$ be reals in the final extension. Exactly one of the two tags*

$$\langle \mathrm{wo}, x, y \rangle, \qquad \langle \mathrm{wo}, y, x \rangle$$

is intentionally coded by a well-order stage.

Proof. Choose α such that $x, y \in W_\alpha$. By bookkeeping, the pair appears at some later stage $\beta \in E_{\mathrm{wo}}$. At that stage the canonical order $<^{\beta}_{\mathrm{can}}$ compares x and y, and the stage codes exactly the orientation selected by that order. If the pair appears at an even later stage, the same orientation is computed by Lemma 4.3. Thus the reverse tag is never intentionally coded, while the correct tag is coded at least once. □

Corollary 4.11. *In the final extension, $<_\Delta$ agrees with the global canonical order obtained from the coherent system $\langle <^{\alpha}_{\mathrm{can}} | \ \alpha < \omega_3 \rangle$. In particular $<_\Delta$ is a well-order of the reals, provided no accidental base codes occur.*

Proof. By Lemma 4.10, the correct orientation of every pair is intentionally coded. By Lemma 3.8, no non-intentional well-order tag becomes Φ-true. Hence Definition 4.7 recovers exactly the coherent canonical order. □

4.5 How the uniformization stages select the least successful triple

We end the section by recording the selection mechanism in the form used in the final verification. Fix an odd formula φ_m and a real x in the final extension. In the final model list all triples (x, y, a_0) by the order $<^{x}_{\mathrm{tr}}$ which puts $(x, 0, 0)$ first and then follows the final $<_\Delta$-order on pairs (y, a_0). Since $<_\Delta$ agrees with the coherent canonical order, every bounded initial segment of this modified list is already the corresponding initial segment in a sufficiently late intermediate model.

If $t = (x, y, a_0)$ is the first triple satisfying $\Theta_m(t)$, then all earlier triples fail. This includes the special zeroth triple $(x, 0, 0)$ if it is not itself the successful triple. Choose witnesses to these failures according to the dual alternating pattern and compress them by the coherent maps π^{β}_{ξ}. For each play of the universal variables in the copied failure test, the remaining witnesses can be chosen and compressed so that the odd copied failure test (O) holds. The bookkeeping therefore presents the corresponding finite tag cofinally often, and at every sufficiently late presentation the stage deliberately codes

$$\langle \mathrm{unif}, \mathrm{odd}, m, x, y, a_0, b_1, \ldots, b_{2n-2} \rangle.$$

Conversely, if t' is later than the first successful triple, then the earlier list already contains a successful triple. For every proposed first compressed witness there is a continuation of the dual play which makes the copied failure test fail at that earlier successful coordinate. Hence the corresponding odd tag is never intentionally coded, and Lemma 3.8 turns this into $\neg\Phi$ in the final model.

The even case is the formal dual. For the least successful triple, the correctly compressed witnesses make the even copied failure test (E) hold, and the stage leaves the associated even tag uncoded. For any later triple, the earlier successful triple makes the test fail along a suitable continuation of the play; by the even stage rule the corresponding tag is then deliberately coded. This is the mechanism behind the appearance of Φ in the odd clauses and $\neg\Phi$ in the even clauses of the final uniformizing predicates.

5 Verification of the final model

Let $G^{\mathrm{pre}} \subseteq \mathbb{P}^{\mathrm{pre}}$ be generic over L^*, put $L^\dagger = L^*[G^{\mathrm{pre}}]$, and let $G \subseteq \mathbb{P}_{\omega_3}$ be generic over $L^\dagger$. For $\alpha \leq \omega_3$, put

$$W_\alpha = L^\dagger[G \cap \mathbb{P}_\alpha].$$

We verify the four conclusions of Theorem 1.1. We shall repeatedly use two elementary consequences of the way the iteration was arranged. First, every real in the final model has bounded support over $L^\dagger$. Secondly, every intentional coding event is attached to a fresh block and no block is ever reused, and Lemma 3.8 converts this stage-by-stage freshness into a final projective truth value.

5.1 Cardinals, supports and the value of the continuum

Lemma 5.1 (Size of the final iteration)**.** *For every $\alpha \leq \omega_3$,*

$$|\mathbb{P}_\alpha| \leq \aleph_3,$$

and $|\mathbb{P}_{\omega_3}| = \aleph_3$. The forcing $\mathbb{P}_{\omega_3}$ is c.c.c. and therefore preserves all cardinals and cofinalities. Moreover the final extension satisfies

$$2^{\aleph_0} \leq \aleph_3.$$

Proof. Every non-trivial iterand is of the form $\mathrm{Code}(\rho, z)$. By Lemma 3.3, such an iterand is σ-centered, hence c.c.c., and it has size at most $\aleph_3$ in the corresponding intermediate model. Thus the finite-support iteration is c.c.c. Since the iteration has length ω_3 and each condition has finite support, the standard induction using $(\aleph_3)^{<\omega} = \aleph_3$ gives $|\mathbb{P}_\alpha| \leq \aleph_3$ for every $\alpha \leq \omega_3$. The equality at the end follows because cofinally many stages are non-trivial and use pairwise fresh coordinates.

The ground model for the main recursion, $L^\dagger$, satisfies $(\aleph_3)^{\aleph_0} = \aleph_3$ and already has continuum $\aleph_3$ by Lemma 4.2. Hence the number of nice $\mathbb{P}_{\omega_3}$-names for reals is at most

$$|\mathbb{P}_{\omega_3}|^{\aleph_0} = \aleph_3.$$

Therefore the final extension has at most $\aleph_3$ reals. $\square$

Lemma 5.2 (Bounded supports)**.** *Every $\mathbb{P}_{\omega_3}$-name for a real is equivalent to a name whose support is bounded in ω_3. More generally, if X is a set of reals of cardinality $< \aleph_3$ in $L^\dagger[G]$, then there is an $\alpha < \omega_3$ such that $X \subseteq W_\alpha$.*

Proof. A nice name for a real is a countable set of pairs $(\check{n}, p)$ with $n < \omega$ and $p \in \mathbb{P}_{\omega_3}$. Since each condition has finite support, the union of the supports of all conditions appearing in the name is countable, hence bounded in the regular cardinal ω_3. The name is therefore equivalent to a $\mathbb{P}_\alpha$-name for some $\alpha < \omega_3$.

For the second assertion, choose in the final model an enumeration $\langle x_\xi \mid \xi < \lambda \rangle$ of X with $\lambda < \omega_3$. For each $\xi < \lambda$, choose a bounded support for a name of x_ξ. Since ω_3 is regular, the union of $< \omega_3$ many bounded subsets of ω_3 is bounded. Thus all members of X belong to a common intermediate model W_α. $\square$

Lemma 5.3. *In $L^\dagger[G]$,*

$$\mathfrak{b} = \mathfrak{c} = \aleph_3.$$

Proof. The inequality $\mathfrak{c} \leq \aleph_3$ is Lemma 5.1. For $\mathfrak{c} \geq \aleph_3$, note that at every stage $\alpha \in E_{\text{dom}}$ the forcing is $\text{Code}(\rho_\alpha, \langle \text{dummy}, 0 \rangle)$. By Lemma 3.3, this forcing adds a dominating real over W_α, hence in particular a real not belonging to W_α. Since E_{dom} is cofinal of order type ω_3 and the blocks are fresh, the final model contains at least $\aleph_3$ distinct reals. Thus $\mathfrak{c} = \aleph_3$.

We next show $\mathfrak{b} \geq \aleph_3$. Let $X \subseteq \omega^\omega$ have size $< \aleph_3$ in the final model. By Lemma 5.2, choose $\alpha < \omega_3$ such that $X \subseteq W_\alpha$. Pick $\beta \in E_{\text{dom}}$ with $\beta > \alpha$. The stage β forcing adds a dominating real d_β over W_β, hence over W_α. Therefore d_β eventually dominates every member of X. Thus no family of size $< \aleph_3$ is unbounded. Since always $\mathfrak{b} \leq \mathfrak{c}$, we get $\mathfrak{b} = \mathfrak{c} = \aleph_3$. $\square$

5.2 The projective well-order

Lemma 5.4. *In $L^\dagger[G]$, the relation $<_\Delta$ from Definition 4.7 is exactly the global canonical order induced by the coherent system $\langle <^\alpha_{\text{can}} \mid \alpha < \omega_3 \rangle$. Consequently it is a well-order of the reals.*

Proof. Let $x \neq y$ be reals in the final model. Choose α with $x, y \in W_\alpha$. By the bookkeeping, the pair (x, y) is listed at cofinally many later stages in E_{wo}. At any such stage β, the stage rule compares x and y by $<^\beta_{\text{can}}$ and intentionally codes exactly the tag corresponding to the smaller-first orientation. By Lemma 4.3, all later appearances of the same pair compute the same orientation. Therefore exactly one of

$$\langle \mathrm{wo}, x, y \rangle, \qquad \langle \mathrm{wo}, y, x \rangle$$

is intentionally coded.

By Lemma 3.8, a well-order tag is Φ-true in the final model if and only if it was intentionally coded at some stage. Hence $x <_\Delta y$ holds exactly when the coherent canonical order puts x before y. Since the latter is a well-order, so is $<_\Delta$. □

Lemma 5.5. *The relation $<_\Delta$ is a Δ^1_3 well-order of the reals of $L^\dagger[G]$.*

Proof. By Definition 4.7,

$$x <_\Delta y \quad \Longleftrightarrow \quad \Phi(\langle \mathrm{wo}, x, y \rangle).$$

The right-hand side is Σ^1_3 by Lemma 3.5. On the other hand, by Lemma 5.4, for all reals x, y,

$$x \not<_\Delta y \quad \Longleftrightarrow \quad x = y \ \text{ or } \ \Phi(\langle \mathrm{wo}, y, x \rangle).$$

The right-hand side is again Σ^1_3. Thus both $<_\Delta$ and its complement are Σ^1_3, so $<_\Delta$ is Δ^1_3. □

5.3 The co-analytic mad family

Lemma 5.6. *In $L^\dagger[G]$ there is a Π^1_1 mad family.*

Proof. We apply the preservation theorem to the combined forcing, not merely to the main iteration over $L^\dagger$. By Lemma 4.2, the prelude is a finite-support iteration of σ-centered s.p.s. forcings, and by Lemma 4.8 the main recursion has the same form over $L^\dagger$. Flattening

$$\mathbb{P}^{\mathrm{tot}} = \mathbb{P}^{\mathrm{pre}} * \dot{\mathbb{P}}^{\mathrm{main}}_{\omega_3}$$

to a single finite-support iteration over L^*, every iterand is σ-centered and has the s.p.s. property. Hence Theorem 2.8 applies directly to $\mathbb{P}^{\mathrm{tot}}$. It gives, already in L^*, an $\aleph_1$-perfect Σ^1_2 mad family

$$\mathcal{A} = \bigcup_{\xi<\omega_1} \mathcal{A}_\xi$$

whose reinterpretation remains mad in $L^*[G^{\mathrm{tot}}] = L^\dagger[G]$. The same theorem also gives that this reinterpretation is still Σ^1_2.

Törnquist's theorem says that the existence of a Σ^1_2 mad family is equivalent to the existence of a Π^1_1 mad family. Therefore the final model contains a co-analytic mad family. □

5.4 Copied minimality predicates and complexity checks

We now turn to uniformization. Projective formulas may have real parameters; as usual, a real parameter is folded into the displayed variable x or into the formula code m. We write

$$\mathsf{C}(z) \quad \text{for} \quad \Phi(z), \qquad \mathsf{N}(z) \quad \text{for} \quad \neg\Phi(z).$$

Thus C is Σ^1_3 and N is Π^1_3 by Lemma 3.5.

Fix a formula code m and a real x. The triples with first coordinate x are ordered by the modified order $<^x_{\mathrm{tr}}$: the first triple is always $(x, 0, 0)$, and the remaining triples are ordered by the final $<_\Delta$-order of the attached pairs (y, a_0). We write $s <_x t$ for this modified order. Since $<_\Delta$ agrees with the coherent canonical order, every bounded initial segment of the final $<_x$-list is already the corresponding initial segment in all sufficiently late intermediate models. The compression maps from Lemma 4.5 are used only at bookkeeping stages: the final projective formulas below mention only the finite tags which were coded or intentionally left uncoded. In particular, if $(x, 0, 0)$ is successful, then it is the least successful triple and the selected uniformizing value is 0.

Odd levels

Suppose

$$\varphi_m(x, y) \equiv \exists a_0\, \forall a_1\, \exists a_2 \cdots \exists a_{2n-2}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-2}),$$

where $n \geq 1$ and $\psi_m \in \Pi^1_2$. For $t = (x, y, a_0)$ set

$$\Theta_m(t) \quad \Longleftrightarrow \quad \forall a_1 \exists a_2 \cdots \exists a_{2n-2}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-2}).$$

Put

$$\ell_O(n) = \max(1, 2n-2).$$

For reals $b_1, \ldots, b_{\ell_O(n)}$ define

$$\mathrm{Tag}^O_m(t, b_1, \ldots, b_{\ell_O(n)}) = \langle \mathrm{unif}, \mathrm{odd}, m, x, y, a_0, b_1, \ldots, b_{\ell_O(n)} \rangle.$$

When $n = 1$, the real b_1 is a dummy compression real; no sequence is decoded from it. Its only role is to leave the same visible finite-tag pattern as in the higher odd levels and to keep the defining formula on the Σ^1_3 level.

Define the odd selection relation by

$$\mathrm{Sel}^O_m(t) \quad \Longleftrightarrow \quad \Theta_m(t) \wedge \begin{cases} \exists b_1\; \mathsf{C}\big(\mathrm{Tag}^O_m(t, b_1)\big), & n = 1, \\ \exists b_1 \forall b_2 \exists b_3 \cdots \forall b_{2n-2}\; \mathsf{C}\big(\mathrm{Tag}^O_m(t, b_1, \ldots, b_{2n-2})\big), & n > 1. \end{cases}$$

Lemma 5.7 (Odd complexity). *The relation*

$$F_m^O(x, y) \iff \exists a_0 \ \mathrm{Sel}_m^O(x, y, a_0)$$

is Σ^1_{2n+1}.

Proof. The success predicate $\Theta_m(t)$ is Π^1_{2n}. After the first existential quantifier over b_1, the tail

$$\forall b_2 \exists b_3 \cdots \forall b_{2n-2} \ \mathsf{C}(\mathrm{Tag}_m^O(t, b_1, \dots, b_{2n-2}))$$

is Π^1_{2n} when $n > 1$: the base predicate C is Σ^1_3, and the displayed alternating block supplies exactly the missing real quantifier blocks. For $n = 1$, the coded tail is simply $\exists b_1 \, \mathsf{C}(\mathrm{Tag}_m^O(t, b_1))$, and since $\mathsf{C}(z) \equiv \exists R \, \Psi(R, z)$ with $\Psi \in \Pi^1_2$, the conjunction with the Π^1_2 success predicate still gives a Σ^1_3 formula.

Thus, after the existential quantifiers over a_0 and b_1, the conjunction of $\Theta_m(t)$ with the inner tail is a conjunction of two Π^1_{2n} statements. The pointclass Π^1_{2n} is closed under finite conjunctions. Therefore the whole formula is Σ^1_{2n+1}. □

Lemma 5.8 (Odd selection). *For every x, the relation $F_m^O(x, y)$ holds exactly for the y-coordinate of the least triple $t = (x, y, a_0)$, in the modified order $<^x_{\mathrm{tr}}$ whose first triple is $(x, 0, 0)$, satisfying $\Theta_m(t)$.*

Proof. Fix x, and write

$$\langle t_\gamma \mid \gamma < \omega_3 \rangle, \qquad t_\gamma = (x, y_\gamma, a_0^\gamma),$$

for the final $<^x_{\mathrm{tr}}$-enumeration of the triples with first coordinate x. Put $I_\gamma = \{t_\eta : \eta < \gamma\}$. By coherence of $<_\Delta$ and of the maps π^α_ξ, whenever the finitely many parameters under consideration are in some sufficiently late W_α, we have

$$I^\alpha_{t_\gamma} = I_\gamma, \qquad \xi^\alpha_{t_\gamma} = \gamma,$$

and old compression reals decode to the same old sequences. Hence, for each fixed γ and finite tuple $\vec{b}$, the bookkeeping, the odd stage rule, Shoenfield absoluteness, and Lemma 3.8 give

$$\mathsf{C}\big(\mathrm{Tag}_m^O(t_\gamma, \vec{b})\big) \iff \vec{b} \text{ passes the odd copied failure test for } t_\gamma. \qquad (*)$$

Indeed, if the test holds, the corresponding tag is presented cofinally often and is then deliberately coded; if it fails, the failure is already visible at every presentation of the tag, and the tag is never deliberately coded.

If there is no γ with $\Theta_m(t_\gamma)$, then no triple is selected, since Θ_m is a conjunct in Sel_m^O. Otherwise let ξ be least such that

$$\Theta_m(t_\xi).$$

Thus $\neg\Theta_m(t_\eta)$ for all $\eta < \xi$.

First suppose $n = 1$. Then $\Theta_m(t) = \psi_m(t)$. Hence

$$\forall \eta < \xi \ \neg\psi_m(t_\eta),$$

so the test (O_1) holds for t_ξ, independently of the dummy real b_1. By $(*)$,

$$\exists b_1 \ \mathsf{C}\big(\mathrm{Tag}^O_m(t_\xi, b_1)\big),$$

and therefore $\mathrm{Sel}^O_m(t_\xi)$.

Now assume $n > 1$. From $\neg\Theta_m(t_\eta)$, $\eta < \xi$, we have

$$\forall \eta < \xi \ \exists a^\eta_1 \forall a^\eta_2 \exists a^\eta_3 \cdots \forall a^\eta_{2n-2} \ \neg\psi_m(t_\eta, a^\eta_1, \ldots, a^\eta_{2n-2}).$$

Equivalently, choosing coordinates simultaneously,

$$\exists \vec{a}_1 \forall \vec{a}_2 \exists \vec{a}_3 \cdots \forall \vec{a}_{2n-2} \ \forall \eta < \xi \ \neg\psi_m(t_\eta, a^\eta_1, \ldots, a^\eta_{2n-2}), \tag{1}$$

where $\vec{a}_i = \langle a^\eta_i \mid \eta < \xi \rangle$. Compress the existential moves in (1), and decode the universal moves, by the coherent map π_ξ. Thus (1) yields

$$\exists b_1 \forall b_2 \exists b_3 \cdots \forall b_{2n-2} \ (O) \text{ holds for } (t_\xi, b_1, \ldots, b_{2n-2}).$$

Using $(*)$, this gives

$$\exists b_1 \forall b_2 \exists b_3 \cdots \forall b_{2n-2} \ \mathsf{C}\big(\mathrm{Tag}^O_m(t_\xi, b_1, \ldots, b_{2n-2})\big).$$

Together with $\Theta_m(t_\xi)$, this proves

$$\mathrm{Sel}^O_m(t_\xi).$$

It remains to exclude all other triples. If $\beta < \xi$, then $\neg\Theta_m(t_\beta)$, hence

$$\neg\,\mathrm{Sel}^O_m(t_\beta).$$

Let $\beta > \xi$. If $n = 1$, then $t_\xi \in I_\beta$ and $\psi_m(t_\xi)$ holds. Thus (O_1) fails for t_β, for every dummy b_1, and $(*)$ gives

$$\forall b_1 \ \mathsf{N}\big(\mathrm{Tag}^O_m(t_\beta, b_1)\big).$$

So $\mathrm{Sel}^O_m(t_\beta)$ fails.

Finally let $\beta > \xi$ and $n > 1$. Fix b_1, and decode

$$\pi^{-1}_\beta(b_1) = \langle a^\eta_1 \mid \eta < \beta \rangle.$$

Since t_ξ is successful,

$$\forall a^\xi_1 \exists a^\xi_2 \forall a^\xi_3 \cdots \exists a^\xi_{2n-2} \ \psi_m(t_\xi, a^\xi_1, \ldots, a^\xi_{2n-2}).$$

Therefore, after the given b_1, choose b_2 so that its ξ-coordinate is a corresponding a_2^ξ; then, for arbitrary b_3, choose b_4 with the required ξ-coordinate, and continue through the alternating pattern. For the resulting continuation,

$$\psi_m(t_\xi, a_1^\xi, \ldots, a_{2n-2}^\xi)$$

holds at the coordinate $t_\xi \in I_\beta$. Hence the copied failure test (O) for t_β fails. By $(*)$,

$$\forall b_1 \exists b_2 \forall b_3 \exists b_4 \cdots \exists b_{2n-2}\, \mathsf{N}\big(\mathrm{Tag}_m^O(t_\beta, b_1, \ldots, b_{2n-2})\big),$$

with the evident shortened pattern when $n = 2$. This is the negation of the coded tail required in $\mathrm{Sel}_m^O(t_\beta)$. Thus no $\beta > \xi$ is selected.

Consequently the only selected triple is t_ξ. Since $F_m^O(x, y)$ is defined by existentially quantifying over the third coordinate of a selected triple,

$$F_m^O(x, y) \iff y = y_\xi,$$

as required. □

Lemma 5.9. *Every Σ^1_{2n+1} set in the plane has a Σ^1_{2n+1} uniformization in $L^\dagger[G]$.*

Proof. Let A be defined by the displayed formula φ_m. Define

$$F_m(x, y) \iff F_m^O(x, y).$$

By Lemma 5.7, F_m is Σ^1_{2n+1}. If $F_m(x, y)$ holds, then $\Theta_m(x, y, a_0)$ holds for some a_0, so $(x, y) \in A$. If x belongs to the projection of A, the $<^x_{\mathrm{tr}}$-least successful triple exists and Lemma 5.8 says that exactly its y-coordinate is selected. In particular, if $(x, 0, 0)$ is successful, the selected value is 0. Hence F_m is a same-level uniformizing graph for A. □

Even levels

Suppose

$$\varphi_m(x, y) \equiv \exists a_0\, \forall a_1\, \exists a_2 \cdots \forall a_{2n-3}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-3}),$$

where $n \geq 2$ and $\psi_m \in \Sigma^1_2$. For $t = (x, y, a_0)$ set

$$\Theta_m(t) \iff \forall a_1 \exists a_2 \cdots \forall a_{2n-3}\, \psi_m(x, y, a_0, a_1, \ldots, a_{2n-3}).$$

For reals $b_1, \ldots, b_{2n-3}$ define

$$\mathrm{Tag}_m^E(t, b_1, \ldots, b_{2n-3}) = \langle \mathrm{unif}, \mathrm{even}, m, x, y, a_0, b_1, \ldots, b_{2n-3} \rangle.$$

The even selection relation is

$$\mathrm{Sel}_m^E(t) \iff \Theta_m(t) \;\wedge\; \exists b_1 \forall b_2 \exists b_3 \cdots \exists b_{2n-3}\, \mathsf{N}\big(\mathrm{Tag}_m^E(t, b_1, \ldots, b_{2n-3})\big).$$

For $n = 2$ this is simply $\Theta_m(t) \wedge \exists b_1\, \mathsf{N}(\mathrm{Tag}_m^E(t, b_1))$.

Lemma 5.10 (Even complexity)**.** *The relation*

$$F_m^E(x,y) \iff \exists a_0 \ \mathrm{Sel}_m^E(x,y,a_0)$$

is Σ^1_{2n}*. Moreover, for each* x*, it selects exactly the least successful triple for the corresponding even formula.*

Proof. First consider complexity. The success predicate $\Theta_m(t)$ is Π^1_{2n-1}. The predicate N is Π^1_3. After the existential quantifier over b_1, the tail

$$\forall b_2 \exists b_3 \cdots \exists b_{2n-3} \ \mathsf{N}(\mathrm{Tag}_m^E(t, b_1, \ldots, b_{2n-3}))$$

is Π^1_{2n-1}. Thus, after the outer existential quantifiers over a_0 and b_1, the matrix is a conjunction of two Π^1_{2n-1} statements. Hence the whole relation is Σ^1_{2n}. This is the explicit reason why the occurrence of $\neg\Phi$ does not raise the complexity: it is a Π^1_3 predicate placed inside a Π^1_{2n-1} tail, and $n \geq 2$.

It remains to prove selection. Let $t_\xi = (x, y_\xi, a_0^\xi)$ be the least triple in $<^x_{\mathrm{tr}}$ satisfying Θ_m. As in the odd case, compress witnesses to the failures of all t_η with $\eta < \xi$. For every play of the universal variables in the even tail, totality of the maps π^α_ξ decodes the proposed compression reals as sequences indexed by the earlier triples. Choose the remaining counter-witnesses and compress them. The even copied failure test (E) holds for the resulting tuple in every sufficiently late intermediate model, and by Shoenfield absoluteness this valuation is absolute to later stages. The even stage rule therefore deliberately leaves

$$\mathrm{Tag}_m^E(t_\xi, b_1, \ldots, b_{2n-3})$$

uncoded. By Lemma 3.8, the final model satisfies the corresponding N statement. Hence $\mathrm{Sel}_m^E(t_\xi)$ holds.

If $\beta < \xi$, then $\Theta_m(t_\beta)$ is false. If $\beta > \xi$, fix an arbitrary first compression real b_1 for t_β and, using totality and coherence, decode the coordinate corresponding to the earlier successful triple t_ξ. Since t_ξ is successful, the remaining witnesses can be chosen so that the matrix ψ_m is true at that earlier coordinate. Compress these choices into the following variables. For this continuation the even copied failure test (E) fails, and the even stage rule therefore deliberately codes the tag. Lemma 3.8 gives the corresponding C statement in the final model. Thus the alternating N-tail required by $\mathrm{Sel}_m^E(t_\beta)$ fails for every $\beta > \xi$. Hence exactly the least successful triple is selected. □

Lemma 5.11. *Every* Σ^1_{2n} *set in the plane has a* Σ^1_{2n} *uniformization in* $L^\dagger[G]$.

Proof. Given the even formula φ_m, define

$$F_m(x,y) \iff F_m^E(x,y).$$

By Lemma 5.10, F_m is Σ^1_{2n} and selects exactly the y-coordinate of the least successful triple, computed with the zeroth-triple convention, over each nonempty section. It is therefore a uniformizing graph of the same pointclass. □

Corollary 5.12. *In $L^\dagger[G]$, the pointclass Σ^1_k has the uniformization property for every $k \geq 2$.*

Proof. For $k = 2$, this is the classical consequence of Kondo's uniformization theorem for co-analytic sets. Indeed, if $A(x, y)$ is Σ^1_2, write it as $\exists z\, B(x, y, z)$ with $B \in \Pi^1_1$, uniformize the co-analytic set B in the variables $(x, (y, z))$, and then project away the auxiliary coordinate z; the resulting graph is Σ^1_2.

For $k \geq 3$, write k either as $2n + 1$ or as $2n$. The odd cases are Lemma 5.9, and the even cases are Lemma 5.11. □

Proof of Theorem 1.1. Start in L and force with the Fischer–Friedman–Khomskii preliminary forcing to obtain L^*. Then force over L^* with the dummy prelude $\mathbb{P}^{\mathrm{pre}}$, obtaining $L^\dagger$, and finally force over $L^\dagger$ with the main finite-support iteration $\mathbb{P}_{\omega_3}$ from Section 4. Equivalently, force over L^* with the combined forcing $\mathbb{P}^{\mathrm{tot}} = \mathbb{P}^{\mathrm{pre}} * \dot{\mathbb{P}}^{\mathrm{main}}_{\omega_3}$. Lemma 5.3 gives $\mathfrak{b} = \mathfrak{c} = \aleph_3$. Lemma 5.5 gives a Δ^1_3 well-order of the reals. Lemma 5.6 gives a Π^1_1 mad family. Corollary 5.12 gives the Σ^1_k-uniformization property for every $k \geq 2$. □

6 Final remarks

The proof above uses the original length-ω_3 preparation from [FFK13], and therefore the theorem is stated with $\mathfrak{b} = \mathfrak{c} = \aleph_3$. A regular-$\kappa$ variant would require a corresponding regular-κ version of the Δ^1_3 well-order together with the co-analytic mad family preservation theorem. The uniformization part itself is insensitive to this particular value of the continuum once the base coding predicate has the same exact no-accidental-code property.

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