A Universe with Π^1_3-Uniformization and a Δ^1_3-Definable Well-Order of the Reals

Stefan Hoffelner[1*]

[1]TU Wien
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Abstract

We construct a generic extension of L in which CH holds, the reals admit a Δ^1_3-definable well-order, and the Π^1_3-uniformization property holds. Thus, at the third projective level, Π^1_3-uniformization is compatible with the existence of a low-complexity projective well-order of the reals, although such a well-order cannot be good in Addison's sense. The construction uses a countable-support iteration of coding forcings which simultaneously preserves the definable well-order and meets all Π^1_3-uniformization requirements.

1 Introduction

This paper explores, at the third projective level, the interaction between two important properties of the real numbers: the Π^1_3-uniformization property and Δ^1_3-definable well-orderings of the reals. We use standard notation and background from descriptive set theory and forcing; see, for example, [27, 30, 24].

To recall, for a set $A \subset \omega^\omega \times \omega^\omega$, we say that a function f is a uniformization (or uniformizing function) of A if f is a partial function $f : \omega^\omega \to \omega^\omega$, with the domain of f being $\text{pr}_1(A)$, and the graph of f being a subset of A.

Definition 1.1. *Throughout this paper, the projective pointclasses Σ^1_n, Π^1_n, and Δ^1_n are understood in the lightface sense unless explicitly stated otherwise. Thus, for $\Gamma \in \{\Sigma^1_n, \Pi^1_n \mid n \in \omega\}$, we say that Γ has the uniformization property if every lightface Γ-set $A \subseteq \omega^\omega \times \omega^\omega$ has a uniformization whose graph is again a lightface Γ-set.*

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The corresponding boldface version follows from the lightface one by the standard relativization argument: if $A(x, y)$ is defined by a projective formula with a real parameter a, apply the lightface uniformization property to the relation on triples (z, x, y) obtained by replacing the parameter a with the variable z, and then restrict the resulting uniformization to the section $z = a$.

J. Addison observed in [2] that a good well-order of the reals, definable by a Δ^1_n-formula, implies the Σ^1_m-uniformization property for every $m \geqslant n$. We use the following standard formulation. A Δ^1_n-definable well-order $<$ of the reals is called a *good* Δ^1_n-well-order if it has order type ω_1 and if the associated initial-segment coding relation $<_I \subseteq (\omega^\omega)^2$ is again Δ^1_n-definable, where

$$x <_I y \Leftrightarrow \{(x)_k \mid k \in \omega\} = \{z \mid z < y\}.$$

Here $x \mapsto ((x)_k : k < \omega)$ denotes a fixed recursive coding of reals by sequences of reals. Thus $x <_I y$ says precisely that x codes the initial segment of the $<$-order below y.

A theorem of Novikov states that, for each $n \in \omega$, the classes Σ^1_n and Π^1_n cannot both have the uniformization property; see also the standard accounts in [27, 30]. Consequently, Π^1_n-uniformization is incompatible with a good Δ^1_n-definable well-order: by Addison's theorem, such a well-order would yield Σ^1_n-uniformization as well. It is therefore natural, already at the third projective level, to ask whether it is consistent with CH that a Δ^1_3-definable well-order of the reals can be combined with the Π^1_3-uniformization property. This paper answers this question affirmatively.

Theorem. *There exists a generic extension of L in which there is a Δ^1_3-definable well-order of the reals, CH holds, and the Π^1_3-uniformization property holds.*

We concentrate on the L-based third-level case. The strategy is designed to have higher-level analogues over canonical inner models with finitely many Woodin cardinals, along the lines of [9]; however, those adaptations require additional machinery and are not part of the formal theorem proved in the present paper and will be worked out in a follow up.

The techniques used to obtain such a universe are based on the forcing constructions from [11, 9, 13], to which this paper owes much. The broader line of work also includes constructions separating or combining projective separation, reduction, and uniformization principles, such as [12, 14, 19, 15, 20, 21]. However, we employ a different coding machine, resulting in a countable support iteration rather than a finite support iteration as in [12], or the product /iteration hybrids with mixed support as in [9], [13] or [21]. This approach leads a streamlined presentation and allows neater proofs. We believe that this new machine is also of independent interest.

Put in a bigger context, this article contributes to the study of the behaviour of separation, reduction and uniformization on the projective hierarchy in the absence of projective determinacy, contrasting with the classical determinacy pattern associated with projective determinacy [28]. Related work on global Σ-uniformization, large continuum constructions, and projective well-orders includes [16, 18, 6, 5, 17]. The coding methods used here are also related in spirit to the stationary-set and nonstationary-ideal coding techniques developed in [10, 22, 23].

2 Preliminaries

The forcings used in this construction are well-known, but we will briefly introduce them and highlight their key properties.

Definition 2.1 ([3]). *For a stationary set $R \subset \omega_1$, the club-shooting forcing through R, denoted $\mathbb{P}_R$, consists of all continuous strictly increasing functions*

$$p : \alpha + 1 \longrightarrow R$$

with $\alpha < \omega_1$. Equivalently, a condition is a countable closed bounded subset of R, written in its increasing enumeration. The order is end-extension: $q \leqslant p$ iff q extends p as a function, equivalently iff $\operatorname{ran}(q)$ end-extends $\operatorname{ran}(p)$.

The club-shooting forcing $\mathbb{P}_R$ serves as a classic example of an *R-proper forcing*; for background on proper forcing and its variants see [1, 8, 24]. A forcing $\mathbb{P}$ is R-proper if for every condition $p \in \mathbb{P}$, every $\theta > 2^{|\mathbb{P}|}$, and every countable $M \prec H(\theta)$ such that $M \cap \omega_1 \in R$ and $p, \mathbb{P} \in M$, there exists a condition $q < p$ that is $(M, \mathbb{P})$-generic. A condition $q \in \mathbb{P}$ is said to be $(M, \mathbb{P})$-generic if $q \Vdash$ "$\dot{G} \cap M$ is an M-generic filter," where $\dot{G}$ is the canonical name for the generic filter. See also [8].

Lemma 2.2. *Let $R \subset \omega_1$ be stationary and co-stationary. Then the club-shooting forcing $\mathbb{P}_R$ generically adds a club through R. Moreover, $\mathbb{P}_R$ is R-proper, ω-distributive, and thus preserves ω_1. Additionally, R and all its stationary subsets remain stationary in the generic extension.*

We will select a family of sets R_β such that we can shoot an arbitrary pattern of clubs through the elements of R_β, with this pattern being readable from the stationarity of the R_β's in the generic extension. It is crucial to recall that for a stationary, co-stationary set $R \subset \omega_1$, R-proper posets can be iterated with countable support, and the result will again be an R-proper forcing. This follows from the well-known results for plain proper forcings (see [8], Theorem 3.9, and the subsequent discussion).

Fact 2.3. *Let $R \subset \omega_1$ be stationary and co-stationary. Suppose $(\mathbb{P}_\alpha : \alpha < \gamma)$ is a countable support iteration, and let $\mathbb{P}_\gamma$ denote the resulting partial order* *obtained through the countable support limit. If at each stage α, $\mathbb{P}_\alpha \Vdash \dot{\mathbb{P}}(\alpha)$ is R-proper, then $\mathbb{P}_\gamma$ is R-proper.*

Once we decide to shoot a club through a stationary, co-stationary subset of ω_1, this club will be contained in all ω_1-preserving outer models. This gives us a robust method of encoding arbitrary information into a suitably chosen sequence of sets, a method used in several definability and coding constructions, for example [10, 17, 22, 23].

Lemma 2.4. *Let $(R_\alpha : \alpha < \omega_1)$ be a partition of ω_1 into $\aleph_1$-many stationary sets, let $r \in 2^{\omega_1}$ be arbitrary, and let $\mathbb{P}$ be a countable support iteration $(\mathbb{P}_\alpha : \alpha < \omega_1)$, defined inductively by:*

$$\mathbb{P}(\alpha) := \dot{\mathbb{P}}_{\omega_1 \backslash R_{2\cdot\alpha}} \text{ if } r(\alpha) = 1,$$

and

$$\mathbb{P}(\alpha) := \dot{\mathbb{P}}_{\omega_1 \backslash R_{(2\cdot\alpha)+1}} \text{ if } r(\alpha) = 0.$$

Let $G \subseteq \mathbb{P}$ be an arbitrary V-generic filter. Then in $V[G]$, for every $\alpha < \omega_1$,

$$r(\alpha) = 1 \text{ if and only if } R_{2\cdot\alpha} \text{ is nonstationary,}$$

and

$$r(\alpha) = 0 \text{ if and only if } R_{(2\cdot\alpha)+1} \text{ is nonstationary.}$$

Proof. Let $G \subseteq \mathbb{P}$ be an arbitrary V-generic filter and fix $\alpha < \omega_1$. We first record the preservation fact used in both cases. We use here the uniqueness of the decomposition of an ordinal as $2\cdot\delta + i$ with $i < 2$. Suppose that $\xi < \omega_1$ is not one of the indices through whose complement the iteration shoots a club. Then every iterand of $\mathbb{P}$ is of the form $\mathbb{P}_{\omega_1 \backslash R_\beta}$ with $\beta \neq \xi$. Since the sets R_ρ form a partition of ω_1, we have $R_\xi \subseteq \omega_1 \backslash R_\beta$ for every such β. Hence each iterand is R_ξ-proper, and by the preservation theorem for countable support iterations of R_ξ-proper forcings, the whole iteration preserves the stationarity of R_ξ.

Now consider the two possible values of the ground-model function r at α. If $r(\alpha) = 1$, then the α-th stage of the iteration is $\mathbb{P}_{\omega_1 \backslash R_{2\cdot\alpha}}$. Thus a club subset of $\omega_1 \backslash R_{2\cdot\alpha}$ is added, and $R_{2\cdot\alpha}$ is nonstationary in $V[G]$. The other index in the pair, $(2 \cdot \alpha) + 1$, is not used at any stage of the iteration; by the preservation observation, $R_{(2\cdot\alpha)+1}$ remains stationary in $V[G]$.

If $r(\alpha) = 0$, then the α-th stage of the iteration is $\mathbb{P}_{\omega_1 \backslash R_{(2\cdot\alpha)+1}}$. Thus a club subset of $\omega_1 \backslash R_{(2\cdot\alpha)+1}$ is added, and $R_{(2\cdot\alpha)+1}$ is nonstationary in $V[G]$. In this case the index $2 \cdot \alpha$ is not used at any stage of the iteration; again by the preservation observation, $R_{2\cdot\alpha}$ remains stationary in $V[G]$.

The two cases prove the displayed equivalences. □

The second forcing technique we employ is the *almost disjoint coding forcing*, introduced by R. Jensen and R. Solovay (see [26]). In this context, we identify subsets of ω with their characteristic functions, and we use the term *reals* both for elements of 2^ω and for subsets of ω, respectively.

Let $\kappa \leqslant 2^{\aleph_0}$, and let $D = \{d_\xi \mid \xi < \kappa\}$ be an almost disjoint family of subsets of ω. Thus the indexing set of the family has the same cardinality as the set of ordinals to be coded. If $X \subseteq \kappa$, the almost disjoint coding forcing $\mathbb{A}_D(X)$ adds a real $x \subseteq \omega$ satisfying

$$\xi \in X \quad \Longleftrightarrow \quad x \cap d_\xi \text{ is finite.}$$

Definition 2.5. *Let $D = \{d_\xi \mid \xi < \kappa\}$ be an almost disjoint family and let $X \subseteq \kappa$. The almost disjoint coding forcing $\mathbb{A}_D(X)$ consists of pairs $(s, F) \in 2^{<\omega} \times [\kappa]^{<\omega}$. We set $(t, H) \leqslant (s, F)$ if and only if:*

1. *$s \subseteq t$ and $F \subseteq H$, and*
2. *for every $\xi \in X \cap F$, $t \cap d_\xi = s \cap d_\xi$.*

For the remainder of this paper we only use the case $\kappa = \omega_1$. We let $D = \{d_\xi \mid \xi < \omega_1\} \in L$ be the definable almost disjoint family of reals obtained by recursively adding the $<_L$-least real almost disjoint from all previously chosen reals. Whenever we use almost disjoint coding forcing, we code relative to this fixed ω_1-sequence D.

The next two forcings we briefly discuss are Jech's forcing for adding a Suslin tree with countable conditions, and the associated forcing that adds a cofinal branch through a Suslin tree S; see [24, 7] and the applications in [10, 13]. Recall that a set-theoretic tree $(S, <)$ is a *Suslin tree* if it is a normal tree of height ω_1 and has no uncountable antichain. Forcing with a Suslin tree S, where conditions are just nodes in S, is a ccc forcing of size $\aleph_1$. Jech's forcing to generically add a Suslin tree is defined as follows:

Definition 2.6. *Let $\mathbb{P}_J$ be the forcing whose conditions are countable, normal trees ordered by end-extension, i.e. $T_1 \leqslant T_2$ if and only if there exists $\alpha \leqslant \text{height}(T_1)$ such that $T_2 = \{t \upharpoonright \alpha \mid t \in T_1\}$.*

It is well-known that $\mathbb{P}_J$ is σ-closed and adds a Suslin tree. In fact, $\mathbb{P}_J$ is forcing equivalent to $\mathbb{C}(\omega_1)$, the forcing for adding a Cohen subset to ω_1 with countable conditions. If $G \subseteq \mathbb{P}_J$ is V-generic and T is the tree added by G, then T has the additional property that, for any Suslin tree S in the ground model, $S \times T$ is Suslin in $V[G]$. This property can be used to develop a robust coding method (see also [10] for further applications).

Lemma 2.7. *Let V be a universe, and let $S \in V$ be a Suslin tree. Suppose $\mathbb{P}_J$ is Jech's forcing for adding a Suslin tree. Let $g \subset \mathbb{P}_J$ be a generic filter, and let $T = \bigcup g$ denote the generic tree. If we let $T \in V[g]$ be the forcing that adds an ω_1-branch b through T, then:*

$$V[g][b] \models S \textit{ is Suslin.}$$

Proof. Let $\dot{T}$ be the $\mathbb{P}_J$-name for the generic Suslin tree. We claim that $\mathbb{P}_J * \dot{T}$ has a dense subset which is σ-closed. Since σ-closed forcings preserve ground model Suslin trees, this will be sufficient. To prove the claim, consider the following set:

$$\{(p, \check{q}) \mid p \in \mathbb{P}_J \wedge \text{height}(p) = \alpha + 1 \wedge \check{q} \text{ is a node of } p \text{ at level } \alpha\}.$$

It is easy to check that this set is dense and σ-closed in $\mathbb{P}_J * \dot{T}$. ☐

We shall use the following direct corollary of the same dense-set argument.

Corollary 2.8. *Let I and B be sets of cardinality at most ω_1, and let $e : B \to I$. Let*

$$\mathbb{R}_I = \prod_{i \in I}^{\text{cs}} \mathbb{P}_J$$

be the countable-support product of Jech forcings, and let $\dot{T}_i$ be the i-th generic Suslin tree added by $\mathbb{R}_I$. In the $\mathbb{R}_I$-extension, put

$$\dot{\mathbb{B}}_e = \prod_{b \in B}^{\text{cs}} \dot{T}_{e(b)}.$$

Then the two-step forcing $\mathbb{R}_I * \dot{\mathbb{B}}_e$ has a dense σ-closed subset. In particular, it adds no reals and preserves every ground-model Suslin tree.

Proof. We describe a dense closed set. A condition belongs to $\mathcal{D}$ if it is of the form $(p, \check{q})$ and there is an ordinal $\alpha < \omega_1$ such that the following hold. First, $p \in \mathbb{R}_I$ has countable support, and for every active coordinate $i \in \text{supp}(p)$, the tree $p(i)$ has height $\alpha + 1$. Second, q is a countable partial function with domain contained in B, and for every $b \in \text{dom}(q)$, the value $q(b)$ is a node on the top level α of the tree $p(e(b))$. We also require $e[\text{dom}(q)] \subseteq \text{supp}(p)$.

The set $\mathcal{D}$ is dense. Given a condition in $\mathbb{R}_I * \dot{\mathbb{B}}_e$, use the σ-closure of the first factor to strengthen the $\mathbb{R}_I$-condition so that it decides the countable support of the second coordinate and all nodes appearing there. Then end-extend the countably many active Jech trees to one common new top level, extending each decided node to that level. This gives a stronger condition in $\mathcal{D}$.

The set $\mathcal{D}$ is σ-closed. Let $(p_n, \check{q}_n)_{n<\omega}$ be a decreasing sequence in $\mathcal{D}$, and let α_n be the common top level of $(p_n, \check{q}_n)$. Let $\alpha = \sup_{n<\omega} \alpha_n < \omega_1$. The union of the supports of the p_n's and of the domains of the q_n's is countable. For each active Jech coordinate, take the union of the increasing sequence of countable trees and add a new top level at height α. For each branch-coordinate b appearing in some q_n, the nodes $q_n(b)$, from the point where they are defined onward, form an increasing chain; put its union on the new top level of the tree at coordinate $e(b)$. The resulting pair $(p_\omega, \check{q}_\omega)$ is a lower bound in $\mathcal{D}$. Thus $\mathcal{D}$ is dense and σ-closed. □

The Suslin trees added by Jech's forcing have an important property that we will exploit. First, we define:

Definition 2.9. *Let T be a Suslin tree. We say that T is* Suslin off the generic branch *if, after forcing with T to add a generic branch b, the tree T_p remains Suslin for every node $p \in T$ that is not in b.*

More generally, let T be Suslin and let $n \in \omega$. We say that T is n-fold Suslin off the generic branch *if, whenever $H \subseteq T^n := \prod_{i<n} T$ is V-generic and H adds the branches $b_0, \dots, b_{n-1}$, the tree T_p remains Suslin in $V[H]$ for any node p that is not in any of the b_i's.*

The next result, due to G. Fuchs and J. Hamkins [7], shows that Suslin trees added by Jech's forcing have this property:

Theorem 2.10. *Let T be a $\mathbb{P}_J$-generic Suslin tree. Then T is* n-fold Suslin off the generic branch *for every $n \in \omega$.*

Iteration notation. For the rest of the paper, all forcing iterations are taken with countable support, unless a different support is explicitly specified. We write

$$\mathop{*}^{\mathrm{cs}}_{i<\delta} \dot{\mathbb{Q}}_i$$

for the countable-support iteration of the iterands $\dot{\mathbb{Q}}_i$. Thus, at a limit stage $\lambda \leqslant \delta$, a condition in the limit forcing has countable support contained in λ, and its i-th coordinate is a $\mathbb{P}_i$-name for a condition in $\dot{\mathbb{Q}}_i$. The symbol $*^{\mathrm{cs}}$ is only notation for this countable-support iteration; finite-support iterations will always be mentioned explicitly when they occur.

Note that if T is n-fold Suslin off the generic branch, and $\{t_i \mid i < \omega\}$ is an antichain in T, then the iteration $*^{\mathrm{cs}}_{i<\omega} T_{t_i}$ is proper. Moreover, if t is a node that is not on any of the generic branches added by $*^{\mathrm{cs}}_{i<\omega} T_{t_i}$, then T_t remains Suslin in the generic extension by $*^{\mathrm{cs}}_{i<\omega} T_{t_i}$ by a result of T. Miyamoyo [29] (see also Theorem 2.15).

Similarly, if T is n-fold Suslin off the generic branch, then any iteration $*^{\mathrm{cs}}_{i<\delta} \check{T}$, for $\delta < \omega_1$, is proper and thus preserves $\aleph_1$. Moreover, the full ω_1-length iteration $*^{\mathrm{cs}}_{i<\omega_1} \check{T}$ is proper.

This can be seen by noting that any condition $\vec{p} \in (*^{\mathrm{cs}}_{i<\delta} \check{T})$ can be strengthened to a condition $\vec{q} < \vec{p}$, such that there exists a set $(\dot{t}_i \mid i < \delta)$, where each $\dot{t}_i$ is a $*^{\mathrm{cs}}_{j<i} T$-name for a node in T, and:

$$1 \Vdash_{*^{\mathrm{cs}}_{i<\delta} T_i} \{\dot{t}_i \mid i < \delta\} \text{ is an antichain in } T.$$

The forcing $*^{\mathrm{cs}}_{i<\delta} T$ below $\vec{q}$ is equivalent to the forcing $*^{\mathrm{cs}}_{i<\delta} T_{\dot{t}_i}$. Since each T is n-fold Suslin off the generic branch, this forcing is a countable support iteration of forcings with the ccc, making it proper. Thus, $*^{\text{cs}}_{i<\delta}\check{T}$ is locally proper, meaning that below any condition there is a stronger condition such that the forcing below it is proper. Local properness implies properness, as locally proper forcings preserve stationarity in $[\lambda]^\omega$ to which properness is equivalent.

We emphasize that we need countable support for this argument. A finite support iteration of $*^{\text{cs}}_{i<\delta}\check{T}$ would collapse $\aleph_1$.

Next, we turn to Jech's forcing $\mathbb{P}_J$ and the product of $\mathbb{P}_J$-forcing. We observe that a product of $\mathbb{P}_J$-forcings will add Suslin trees, and we can destroy these trees without unwanted interference.

Lemma 2.11. *Let S be a Suslin tree in V, and let $\mathbb{P}$ be a countably supported product of length ω_1 of forcings $\mathbb{P}_J$, with G as its generic filter. Then in $V[G]$, there is an ω_1-sequence of Suslin trees $\vec{T} = (T_\alpha \mid \alpha \in \omega_1)$ such that for any finite $e \subset \omega$ with pairwise different members, the tree $S \times \prod_{i \in e} T_i$ will be a Suslin tree in $V[G]$.*

We define a sequence of Suslin trees as follows:

Definition 2.12. *Let $\vec{T} = (T_\alpha \mid \alpha < \kappa)$ be a sequence of Suslin trees. We say that the sequence is an* independent family of Suslin trees *if, for every finite set of pairwise distinct indices $e = \{e_0, e_1, \ldots, e_n\} \subset \kappa$, the product $T_{e_0} \times T_{e_1} \times \cdots \times T_{e_n}$ is a Suslin tree again.*

We summarize the previous results as follows:

Theorem 2.13. *Let $\mathbb{P}$ be the countably supported product of Jech's forcing $\mathbb{P}_J$. Then $\mathbb{P}$ adds an ω_1-sequence $\vec{S} = (S_\alpha \mid \alpha < \omega_1)$ of independent Suslin trees with the following properties:*

1. *Every $S_\alpha \in \vec{S}$ is n-fold Suslin off the generic branch for every $n \in \omega$,*
2. *For every $\alpha < \omega_1$, every $\delta < \omega_1$, and every S_α-antichain of nodes $\{s_i \mid i < \delta\} \subset S_\alpha$, the set:*

$$(\vec{S} \backslash \{S_\alpha\}) \cup \{(S_\alpha)_{s_i} \mid i < \delta\}$$

is an independent set of Suslin trees.

2.1 The ground model W of the iteration

We have to first create a suitable ground model W over which the actual iteration will take place. W will be a generic extension of L which has no new reals. Moreover W has the crucial property that in W there is an ω_1-sequence $\vec{S}$ of ω_1 trees which is $\Sigma_1(\{\omega_1\})$-definable over $H(\omega_2)^W$ (i.e. the definiton is a Σ_1-formula with ω_1 as its only parameter) and which forms an independent sequence of Suslin trees in an inner model of W. The sequence

$\vec{S}$ will enable a coding method we will use throughout this article all the time.

To form W, we start with Gödels constructible universe L as our ground model. We first fix an appropriate sequence of stationary, co-stationary subsets of ω_1 as follows. We use the following standard definability feature of Jensen's canonical construction of $\diamondsuit$ in L. There is a canonical $\diamondsuit_{\omega_1}$-sequence

$$\vec{a} = \langle a_\alpha : \alpha < \omega_1 \rangle$$

in L such that the relation "$\xi \in a_\alpha$" is Σ_1-definable, as a class, over L_{ω_1}; this is the sequence obtained from Jensen's fine-structural construction of $\diamondsuit$ in L (see [25]). We fix this sequence throughout. Thus, for every $A \subseteq \omega_1$, the set

$$\{\alpha < \omega_1 : a_\alpha = A \cap \alpha\}$$

is stationary.

The canonical $\diamondsuit$-sequence can be used to produce an easily definable sequence of stationary, co-stationary subsets. We list the reals in L in the canonical $<_L$-increasing ω_1-sequence $(r_\xi : \xi < \omega_1)$. For each $\xi < \omega_1$, let $\tilde{r}_\xi \in 2^{\omega_1}$ be the function which agrees with r_ξ on the first ω coordinates and is constantly 0 on $[\omega, \omega_1)$. We then write

$$\widehat{r}_\xi := \{\eta < \omega_1 : \tilde{r}_\xi(\eta) = 1\}$$

for the subset of ω_1 coded by this characteristic function. Now, for every $\beta < \omega_1$, define

$$R'_\beta = \{\alpha < \omega_1 : a_\alpha = \widehat{r}_\beta \cap \alpha\}.$$

It is clear that $\forall \alpha \neq \beta (R'_\alpha \cap R'_\beta \in \mathrm{NS}_{\omega_1})$ and we obtain a sequence of pairwise disjoint stationary sets as usual via setting for every $\beta < \omega_1$

$$R_\beta = R'_\beta \backslash \bigcup_{\alpha<\beta} R'_\alpha.$$

Let $\vec{R} = (R_\alpha : \alpha < \omega_1)$. Via picking out one element of $\vec{R}$ and re-indexing we assume without loss of generality that there is a stationary, co-stationary $R \subset \omega_1$, which has pairwise empty intersection with every $R_\beta \in \vec{R}$. The definability point needed later is that the sequence $\vec{R}$ has the same low complexity as the canonical objects from which it was built. The canonical $\diamondsuit$-sequence $\vec{a}$, the $<_L$-enumeration of the reals of L, and the map $(\xi, \alpha) \mapsto \widehat{r}_\xi \cap \alpha$ are uniform over L_{ω_1}; moreover, the passage from R'_β to the disjoint refinement R_β is only a bounded minimization over $\xi \leqslant \beta$. Hence membership in R_β is uniformly Σ_1-definable over the model L_{ω_1}, i.e. there is a Σ_1-formula $\psi(x, y)$ such that for every $\beta < \omega_1$

$$\alpha \in R_\beta \iff L_{\omega_1} \models \psi(\alpha, \beta).$$

We proceed with adding $\aleph_1$-many Suslin trees using of Jech's Forcing $\mathbb{P}_J$. We let

$$\mathbb{Q}^0 = \prod_{\beta \in \omega_1} \mathbb{P}_J$$

using countable support. This is a σ-closed, hence proper notion of forcing. We denote the generic filter of $\mathbb{Q}^0$ with $\vec{S} = (S_\alpha : \alpha < \omega_1)$ and note that by Lemma 2.11 $\vec{S}$ forms an independent family of Suslin trees. We fix, once and for all, the $<_L$-least bijection $e : [\omega_1]^\omega \to \omega_1$ in the global well-order of L. We identify the trees in $(S_\alpha : \alpha < \omega_1)$ with their images under e, so the trees will always be subsets of ω_1 from now on.

We shall single out the first tree of $\vec{S}$, as later we will use this tree in a different way than all the other trees, namely to generically produce sets of indices where some coding will take place. For this reason we will re-index $\vec{S} = (S_\alpha \mid \alpha < \omega_1)$ in defining $S'_{-1} := S_0$ and $\vec{S}' := (S_\alpha \mid 1 \leqslant \alpha < \omega_1)$. To ease notation we will write $\vec{S}$ for the just defined $\vec{S}'$ again. That is

$$\vec{S} = (S_\alpha \mid \alpha < \omega_1)$$

is an independent sequence of Suslin trees and S_{-1} is another Suslin tree such that

$$\vec{S} \cup S_{-1}$$

still forms an independent sequence.

We next define the second block of forcing over $L[\mathbb{Q}^0]$. In this paragraph the sequence $\vec{S} = (S_\beta \mid \beta < \omega_1)$ has already been added by the first block, while the distinguished tree S_{-1} is not a member of this sequence. We define

$$\mathbb{Q}^1 = \prod_{\beta < \omega_1}^{\text{cs}} S_\beta$$

to be the countable-support product of the tree forcings S_β, $\beta < \omega_1$. Thus a condition in $\mathbb{Q}^1$ is a function p with countable domain $\text{dom}(p) \subseteq \omega_1$ such that $p(\beta) \in S_\beta$ for every $\beta \in \text{dom}(p)$. If $\beta \notin \text{dom}(p)$, we regard the β-th coordinate as trivial. The order is coordinatewise extension: $q \leqslant p$ if $\text{dom}(p) \subseteq \text{dom}(q)$ and, for every $\beta \in \text{dom}(p)$, the node $q(\beta)$ extends $p(\beta)$ in S_β. There is no coordinate for S_{-1} in $\mathbb{Q}^1$. If $G^1 \subseteq \mathbb{Q}^1$ is generic over $L[\mathbb{Q}^0]$, then G^1 induces a cofinal branch b_β through each S_β for $\beta < \omega_1$.

By Corollary 2.8, applied with $I = B = \omega_1$ and e the identity map, the two-step forcing $\mathbb{Q}^0 * \dot{\mathbb{Q}}^1$ has a dense σ-closed subset. Hence $L[\mathbb{Q}^0][G^1]$ is a proper and ω-distributive generic extension of L; in particular, no reals are added by the first two forcing blocks.

In a third step we code the trees from $\vec{S} \cup S_{-1}$ into the sequence of L-stationary subsets $\vec{R}$ we produced earlier, using Lemma 2.4. It is important to note that this third forcing block is defined after the $\mathbb{Q}^0$-generic sequence of trees has been added, but before and independently of the $\mathbb{Q}^1$-generic branches.

More formally, over L we regard the third forcing as a $\mathbb{Q}^0$-name $\dot{\mathbb{Q}}^2$. If $G^0 \subseteq \mathbb{Q}^0$ is L-generic and $\vec{S} \cup \{S_{-1}\}$ is the corresponding sequence of trees in $L[G^0]$, then

$$\mathbb{Q}^2 = (\dot{\mathbb{Q}}^2)^{G^0}$$

is the forcing defined below in $L[G^0]$. No $\mathbb{Q}^1$-generic branch is used in this definition. Equivalently, if $G^1 \subseteq \mathbb{Q}^1$ is any $\mathbb{Q}^1$-generic filter over $L[G^0]$, then applying the same recursive definition in $L[G^0][G^1]$ gives the same partial order. This is because the two-step forcing $\mathbb{Q}^0 * \dot{\mathbb{Q}}^1$ is ω-distributive, and the parameters from which $\mathbb{Q}^2$ is defined, namely $\vec{S} \cup \{S_{-1}\}$, $\vec{R}$, and ω_1, are the same in $L[G^0]$ and in $L[G^0][G^1]$.

We emphasize the precise meaning of the product notation used below. After the $\mathbb{Q}^0$-generic has been fixed, $\mathbb{Q}^1$ and $\mathbb{Q}^2$ are both partial orders in $L[G^0]$, and the $\mathbb{Q}^2$ obtained by defining it in $L[G^0]$ is literally the same partial order as the one obtained by applying the same definition in $L[G^0][G^1]$. Thus, when we write

$$\mathbb{Q}^0 * (\dot{\mathbb{Q}}^1 \times \dot{\mathbb{Q}}^2),$$

we are not comparing two different definitions of $\mathbb{Q}^2$ up to isomorphism or forcing equivalence. We are using that, once interpreted in $L[G^0]$, the last two forcing blocks are the actual product factors $\mathbb{Q}^1$ and $\mathbb{Q}^2$.

The forcing $\mathbb{Q}^2$ itself is a countable support iteration of length $\omega_1 \cdot \omega_1$ whose components are countable support iterations. Here and below, when we say that such a bijection is definable, we mean that we take the canonical object selected by the global well-order of L. More precisely, let h be the $<_L$-least bijection from $\omega_1 \times \omega_1$ onto $\omega_1 \cup \{-1\}$. Then $h \in L_{\omega_2}$. Using h, we rewrite $\vec{R}$ from now on in order type $\omega_1 \cdot \omega_1$, so we assume that $\vec{R} = (R_\alpha : \alpha < \omega_1 \cdot \omega_1)$. We let $\alpha \in \omega_1 \cup \{-1\}$ and consider the tree $S_\alpha \subset \omega_1$. Defining the α-th factor of our iteration $\mathbb{Q}^2$, we let $\mathbb{Q}^2(\alpha)$ be the countable support iteration which codes the characteristic function of S_α into the α-th ω_1-block of the R_β's just as in Lemma 2.4. So $\mathbb{Q}^2(\alpha)$ is a countable support iteration whose factors, denoted by $\mathbb{Q}^2(\alpha)(\gamma)$, are defined in $L[G^0]$ by

$$\mathbb{Q}^2(\alpha)(\gamma) = \dot{\mathbb{P}}_{\omega_1 \backslash R_{\omega_1 \cdot \alpha + 2\gamma + 1}} \text{ if } S_\alpha(\gamma) = 0$$

and

$$\mathbb{Q}^2(\alpha)(\gamma) = \dot{\mathbb{P}}_{\omega_1 \backslash R_{\omega_1 \cdot \alpha + 2\gamma}} \text{ if } S_\alpha(\gamma) = 1.$$

Recall that we let R be a stationary, co-stationary subset of ω_1 which is disjoint from all the R_α's which are used. It follows from Lemma 2.4 that for every $\alpha \in \omega_1 \cup \{-1\}$, $\mathbb{Q}^2(\alpha)$ is an R-proper forcing which additionally is ω-distributive. Then we let $\mathbb{Q}^2$ be the countably supported iteration,

$$\mathbb{Q}^2 = \mathop{\overset{\text{cs}}{*}}_{\alpha \in \omega_1 \cup \{-1\}} \mathbb{Q}^2(\alpha)$$

which is again R-proper (and ω-distributive as we shall see later). This way we can turn the generically added sequence of trees $\vec{S}$ into a definable sequence of trees. Indeed, if we work in $L[\vec{S} * \vec{b} * G]$, where $\vec{S} * \vec{b} * G$ is $\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2$-generic over L, then, as seen in Lemma 2.4

$$\forall \alpha, \gamma < \omega_1 (\gamma \in S_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma} \text{ is not stationary and}$$
$$\gamma \notin S_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma + 1} \text{ is not stationary})$$

Note here that the above formula can be written in a $\Sigma_1(\{\omega_1\})$-way (i.e. it can be written as a Σ_1 formula with the ordinal ω_1 as the only parameter), as it reflects down to $\aleph_1$-sized, transitive models of ZF^- which contain a club through exactly one element of every pair $\{(R_\alpha, R_{\alpha+1}) : \alpha < \omega_1\}$.

Our goal is to use $\vec{S}$ for coding. For this it is essential that the sequence remains independent in the inner universe obtained by using the $\mathbb{Q}^0$-generic together with the $\mathbb{Q}^2$-generic, but not the $\mathbb{Q}^1$-generic branches. The preceding convention justifies this inner-model notation: after the $\mathbb{Q}^0$-generic has been fixed, the remaining forcing is the actual product $\mathbb{Q}^1 \times \mathbb{Q}^2$, where $\mathbb{Q}^2$ is the same poset whether it is defined in $L[G^0]$ or in $L[G^0][G^1]$. Hence one may form the intermediate model $L[G^0][G^2]$, equivalently denoted below by $L[\mathbb{Q}^0][\mathbb{Q}^2]$, without first adjoining the $\mathbb{Q}^1$-generic branches.

The following line of reasoning is similar to [10]. Recall that for a forcing $\mathbb{P}$ and $M \prec H(\theta)$, a condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic iff for every maximal antichain $A \subset \mathbb{P}$, $A \in M$, it is true that $A \cap M$ is predense below q. The key fact is the following (see [29] for the case where $\mathbb{P}$ is proper, see e.g. [9] or [13] for a proof)

Lemma 2.14. *Let T be a Suslin tree, $R \subset \omega_1$ stationary and $\mathbb{P}$ an R-proper poset. Let θ be a sufficiently large cardinal. Then the following are equivalent:*

1. *$\Vdash_{\mathbb{P}} T$ is Suslin*
2. *if $M \prec H_\theta$ is countable, $\eta = M \cap \omega_1 \in R$, and $\mathbb{P}$ and T are in M, further if $p \in \mathbb{P} \cap M$, then there is a condition $q < p$ such that for every condition $t \in T_\eta$, (q, t) is $(M, \mathbb{P} \times T)$-generic.*

We shall use the following R-proper version of Miyamoto's preservation theorem. The proof is the standard proof of Miyamoto's theorem, with the elementary submodels restricted to those whose intersection with ω_1 lies in R.

Theorem 2.15 (Miyamoto's preservation theorem, R-proper form)**.** *Let $R \subseteq \omega_1$ be stationary, and let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \eta \rangle$$

be a countable-support iteration. Suppose that for every $\alpha < \eta$,

$$\mathbb{P}_\alpha \Vdash \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is } R\textit{-proper and preserves every Suslin tree.''}$$

Then $\mathbb{P}_\eta$ *is R-proper. Moreover, if* T *is a Suslin tree in the ground model, then*

$$\mathbb{P}_\eta \Vdash \text{“}T \text{ is Suslin.''}$$

Proof. The R-properness of $\mathbb{P}_\eta$ is the usual countable-support iteration theorem for R-proper forcings. We include the preservation argument, since this is the point at which the quoted form differs from Miyamoto's theorem for proper iterations.

Fix a ground-model Suslin tree T. By Lemma 2.14, it is enough to verify the corresponding R-proper preservation criterion. Thus let θ be sufficiently large, let

$$M \prec H_\theta$$

be countable with

$$R,\ T,\ \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \eta\rangle \in M \quad \text{and} \quad \delta := M \cap \omega_1 \in R,$$

and let $p \in \mathbb{P}_\eta \cap M$. We show, by induction on η, that there is $q \leqslant p$ such that for every $t \in T_\delta$,

$$(q,t) \text{ is } (M, \mathbb{P}_\eta \times T)\text{-generic.}$$

This gives $\mathbb{P}_\eta \Vdash$ “T is Suslin'' by Lemma 2.14.

The successor step is the standard two-step argument. Suppose $\eta = \beta+1$. By the induction hypothesis applied to $\mathbb{P}_\beta$, strengthen $p \restriction \beta$ to a condition $q_0 \in \mathbb{P}_\beta$ such that for every $t \in T_\delta$,

$$(q_0, t)$$

is $(M, \mathbb{P}_\beta \times T)$-generic. Work below q_0. In the corresponding $\mathbb{P}_\beta$-generic extension, the model $M[G_\beta]$ is still countable and satisfies

$$M[G_\beta] \cap \omega_1 = M \cap \omega_1 = \delta \in R.$$

Moreover, by the induction hypothesis, T remains Suslin after forcing with $\mathbb{P}_\beta$, and by assumption $\dot{\mathbb{Q}}_\beta$ is forced to be R-proper and to preserve Suslin trees. Hence, applying Lemma 2.14 in the $\mathbb{P}_\beta$-extension to the model $M[G_\beta]$, we may choose a $\mathbb{P}_\beta$-name $\dot{q}_1$ such that

$$q_0 \Vdash \text{“}\dot{q}_1 \leqslant p(\beta) \text{ and for every } t \in T_\delta, (\dot{q}_1, t) \text{ is } (M[\dot{G}_\beta], \dot{\mathbb{Q}}_\beta \times T)\text{-generic.''}$$

Then $q = q_0{}^\frown\dot{q}_1$ has the required property for $\mathbb{P}_{\beta+1}$. Indeed, every dense subset of $\mathbb{P}_{\beta+1} \times T$ belonging to M is interpreted, after the first step, as a dense subset of $\dot{\mathbb{Q}}_\beta \times T$ belonging to $M[\dot{G}_\beta]$, and the choice of $\dot{q}_1$ meets it below q_0.

At limit stages, the proof is the usual countable-support fusion argument. Let η be a limit ordinal and let $\langle \eta_n : n < \omega\rangle \in M$ be increasing and cofinal in $M \cap \eta$. Enumerate in M the dense subsets of $\mathbb{P}_\eta \times T$ as $\langle D_n : n < \omega \rangle$. Recursively construct a decreasing sequence of conditions, using the induction hypothesis at the stages η_n, so that the n-th condition meets the projection of D_n and remains $(M, \mathbb{P}_{\eta_n} \times T)$-generic, uniformly for all $t \in T_\delta$. Taking the countable-support limit of this fusion sequence gives a condition $q \leqslant p$. By construction, for every $n < \omega$ and every $t \in T_\delta$, the set $D_n \cap M$ is predense below (q, t). Hence (q, t) is $(M, \mathbb{P}_\eta \times T)$-generic for every $t \in T_\delta$.

The only change from Miyamoto's proof is that throughout the argument the elementary submodels are restricted to those M with $M \cap \omega_1 \in R$. This restriction is preserved at successor stages because $M[\dot{G}_\beta] \cap \omega_1 = M \cap \omega_1$, and it is exactly the class of models to which R-properness applies. □

So in order to argue that our forcing $\mathbb{Q}^2$ preserves Suslin trees when used over the ground model $L[\mathbb{Q}^0]$, it is sufficient to show that every factor preserves Suslin trees. This is indeed the case.

Lemma 2.16. *Let $R \subset \omega_1$ be stationary and co-stationary. Then the club-shooting forcing $\mathbb{P}_R$ preserves Suslin trees.*

Proof. This is exactly [9, Lemma 2.14]. The proof applies Lemma 2.14: for a countable $M \prec H_\theta$ with $M \cap \omega_1 \in R$, one constructs an ω-sequence of stronger conditions meeting the relevant dense sets in the models obtained by adjoining nodes on the $M \cap \omega_1$-level of the Suslin tree. Since the domains of the conditions converge to $M \cap \omega_1 \in R$, their union together with this endpoint is again a condition in $\mathbb{P}_R$, giving the required simultaneous $(M, \mathbb{P}_R \times T)$-genericity. □

Let us set $W := L^{\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2}$ which will serve as our ground model for a second iteration of length ω_1. To summarize the above:

Theorem 2.17. *The universe $W = L^{\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2}$ is an ω-distributive generic extension of L, in particular no new reals are added and ω_1 is preserved. Moreover the following statements hold in W:*

1. *There is a $\Sigma_1(\{\omega_1\})$-definable sequence $\vec{S} \cup \{S_{-1}\}$ which forms an independent family of Suslin trees in the inner model $L^{\mathbb{Q}^0 * \mathbb{Q}^2}$, yet no tree in this sequence, except S_{-1}, is Suslin in W.*
2. *The tree S_{-1} is n-fold Suslin off the generic branch for every $n \in \omega$.*
3. *For every antichain $\{s_i \in S \mid i < \delta\} \subset S_{-1}$, the set $\vec{S} \cup \{(S_{-1})_{s_i} \mid i < \delta\}$ forms an independent family of Suslin trees in $L^{\mathbb{Q}^0 * \mathbb{Q}^2}$.*

Proof. To see that W has no new reals uses a standard argument. As $\mathbb{Q}^0 * \mathbb{Q}^1$ does not add any reals, it is sufficient to show that $\mathbb{Q}^2$ is ω-distributive in $L^{\mathbb{Q}^0 * \mathbb{Q}^1}$. We spell out the fusion step because conditions in the later coordinates of an iteration are names. If p is a condition in $\mathbb{Q}^2$ and $\xi \in$

supp(p), then $p(\xi)$ is a $\mathbb{Q}^2_\xi$-name for a condition in the club-shooting forcing used at coordinate ξ. Thus, whenever we take unions of such conditions, the union is meant coordinatewise and at coordinate ξ it is the canonical $\mathbb{Q}^2_\xi$-name for the union of the interpreted club-shooting conditions.

Let $p \in \mathbb{Q}^2$ and assume that $p \Vdash$ "$\dot{r}$ is a countable sequence of ordinals". We shall find a stronger condition deciding $\dot{r}$. Let $M \prec H(\omega_3)$ be countable with $p, \mathbb{Q}^2, \dot{r} \in M$ and $\delta := M \cap \omega_1 \in R$, where R is the fixed stationary set disjoint from all stationary sets used for coding. Inside M construct a decreasing sequence $(p_n : n < \omega)$ of conditions below p such that p_n decides $\dot{r}(n)$. Let $a(n)$ be the ordinal decided by p_n.

Put
$$A = \bigcup_{n<\omega} \operatorname{supp}(p_n).$$
For $\xi \in A$, the sequence of $\mathbb{Q}^2_\xi$-names $(p_n(\xi) : n < \omega, \xi \in \operatorname{supp}(p_n))$ is forced to be an end-extension decreasing sequence of club-shooting conditions. Let $\dot{u}_\xi$ be the canonical $\mathbb{Q}^2_\xi$-name for the union of their interpretations. Since the coordinate forcing shoots a club through $\omega_1 \backslash R_\rho$ for some coding set R_ρ, and since $\delta \in R \subseteq \omega_1 \backslash R_\rho$, the increasing enumeration of
$$\operatorname{ran}(\dot{u}_\xi) \cup \{\delta\}$$
is forced to be a condition extending all the interpreted conditions $p_n(\xi)$. Let $q(\xi)$ be the corresponding canonical $\mathbb{Q}^2_\xi$-name. For $\xi \notin A$, set $q(\xi) = 1$. Then $q = (q(\xi) : \xi < \omega_1)$ has countable support A and is a condition in $\mathbb{Q}^2$. Moreover $q \leqslant p_n$ for every $n < \omega$, and therefore $q \Vdash \dot{r}(n) = \check{a}(n)$ for all $n < \omega$. Hence $q \Vdash \dot{r} = \check{a}$, as desired.

The remaining assertions follow from the preceding construction and the preservation arguments discussed above. $\square$

Lemma 2.18. *Let S be n-fold Suslin off the generic branch for every $n \in \omega$, and let $\delta < \omega_1$. Let*
$$\langle \mathbb{P}_i, \dot{\mathbb{R}}_i : i < \delta \rangle$$
be a countable-support iteration. Assume that, for every $i < \delta$, one of the following two alternatives holds.

1. $\mathbb{P}_i \Vdash$ "$\dot{\mathbb{R}}_i$ *is proper and preserves Suslin trees."*
2. $\dot{\mathbb{R}}_i$ *is the two-step iterand* $\check{S} * \dot{\mathbb{Q}}_i$, *and*
$$\mathbb{P}_i * \check{S} \Vdash \text{“}\dot{\mathbb{Q}}_i \textit{ is proper and preserves Suslin trees.''}$$

Then $\mathbb{P}_\delta$ is proper.

Proof. This is a direct modification of the argument following Theorem 2.10. There it is shown that the countable-support iteration $*^{\mathrm{cs}}_{i<\delta}\check{S}$ is proper: below any condition it can be strengthened so that the remaining forcing is equivalent to an iteration of cones $S_{\dot{t}_i}$, where the nodes $\dot{t}_i$ form an antichain in S; the n-fold Suslin-off-the-generic-branch property then gives local ccc, hence properness. We use the same argument with arbitrary proper Suslin-tree-preserving forcings inserted between the possible S-steps.

We prove the statement by induction on δ, using the following slightly stronger inductive invariant: after any initial segment of the iteration, the forcing $\check{S}$ is locally ccc. Equivalently, in the corresponding generic extension, every condition in S has a stronger condition t such that the cone S_t is still Suslin. This invariant is what is needed to see that a later occurrence of the S-factor is proper.

At successor stages there are two cases. If the next iterand is of the first kind, then it is forced to be proper, and the usual two-step preservation theorem for proper forcing (see, for example, Abraham [1, Theorem 2.10]) shows that the extended iteration is proper. Moreover, since this iterand is forced to preserve Suslin trees, the local-ccc invariant for S is preserved.

Suppose next that the next iterand is of the form $\check{S}*\dot{\mathbb{Q}}_i$. By the inductive invariant, $\mathbb{P}_i$ forces that $\check{S}$ is locally ccc, hence proper. After forcing with S, the newly added branch is distinct from the branches through S added at earlier S-stages. Thus, below any given condition of S, one can strengthen to a node which is not on any of the already added generic branches through S. Since S is n-fold Suslin off the generic branch for every $n<\omega$ and the intervening forcings preserve Suslin trees, the corresponding cone remains Suslin. Hence the local-ccc invariant is preserved after the S-step. Finally, by assumption, $\mathbb{P}_i * \check{S}$ forces that $\dot{\mathbb{Q}}_i$ is proper and preserves Suslin trees, so the full successor step is proper and again preserves the invariant.

At limit stages we use countable support. Since $\delta<\omega_1$, the limit is a countable-support limit of a countable iteration of proper forcings, and the standard fusion proof for proper countable-support iterations applies. The same fusion preserves the local-ccc invariant for S, because the support is countable and at each finite stage the relevant cone below a sufficiently strong node is a Suslin tree preserved by the intervening iterands. □

We also note that proper forcings of a certain type will leave the sequence $\vec{S}=(S_\alpha \mid \alpha<\omega_1)$ independent.

Lemma 2.19. *Let $\vec{S}*\vec{b}\times G$ be an L-generic filter for $\mathbb{Q}^0*\mathbb{Q}^1\times\mathbb{Q}^2$. Let $A\subset\omega_1$, $A\in L$, and let $\vec{b}\upharpoonright A=(b_\alpha\mid\alpha\in A)$. Work in $W':=L[\vec{S}*(\vec{b}\upharpoonright A)\times G]$. Let $\delta<\omega_1$, and let $\mathbb{P}=(\mathbb{P}_\alpha,\mathbb{P}(\alpha)\mid\alpha<\delta)\in W'$ be a countable support iteration such that for every factor $\mathbb{P}(\alpha)$, either $\mathbb{P}(\alpha)$ is a $\mathbb{P}_\alpha$-name of S_{-1}, or $\mathbb{P}(\alpha)$ is a $\mathbb{P}_\alpha$-name of a proper, $\aleph_1$-sized forcing which preserves Suslin trees.*

Let $\beta\notin A$. Then $S_\beta\in\vec{S}$ remains a Suslin tree in $W'^{\mathbb{P}}$.

Proof. The forcing $\mathbb{P}$ preserves $\aleph_1$ by the last lemma.

We show the assertion via induction on the length of $\mathbb{P}$. If the length is 1, the lemma follows from the fact that $\vec{S} \cup \{S_{-1}\}$ forms an independent family of Suslin trees.

Assume that the lemma is true for iterations of length δ and we want to show it for iterations $\mathbb{P}$ of length $\delta + 1$. If $\mathbb{P} = \mathbb{P}_\delta * \mathbb{P}(\delta)$ and $\mathbb{P}(\delta)$ is the name of a Suslin tree preserving forcing then S_β remains Suslin in $W'^{\mathbb{P}_\delta}$ by the induction hypothesis and hence also in $W'^{\mathbb{P}_\delta * \mathbb{P}(\delta)} = W'^{\mathbb{P}}$, as desired.

So we assume that $\mathbb{P}(\delta)^{G_\delta} = S_{-1}$. The generic branch b through S_{-1} will be different from all the generic branches we added with $\mathbb{P}_\delta$. Thus there is a node $s_{\delta+1}$ such that $s_{\delta+1} \in b$ and $s_{\delta+1}$ is not in any of the $\mathbb{P}_\delta$ added S_{-1}-branches. As S_{-1} is n-fold Suslin off the generic branch for every n, forcing at stage $\delta + 1$ with S below the condition $s_{\delta+1}$ is a ccc forcing over $W^{\mathbb{P}_\delta}$ and also ccc over $W'^{\mathbb{P}_\delta}$ and, by Theorem 2.17, forcing with S_{-1} below $s_{\delta+1}$ preserves ground model Suslin trees, so in particular it preserves S_β.

Finally suppose that $\lambda < \omega_1$ is a limit ordinal and that the assertion has been proved for all shorter iterations. Let $\mathbb{P}_\lambda$ be the countable-support limit of $\langle \mathbb{P}_\xi : \xi < \lambda \rangle$. If $\dot{A}$ is a $\mathbb{P}_\lambda$-name for a maximal antichain of S_β, then $\dot{A}$ is read by a countable set of coordinates, hence by some countable subiteration of $\mathbb{P}_\lambda$. Since λ is countable, the usual countable-support fusion argument for proper iterations applies along an increasing cofinal sequence in λ. At each successor step the induction hypothesis says that S_β is still Suslin, and the next iterand either preserves Suslin trees by assumption or is the S_{-1}-case handled above. The fusion therefore produces a condition forcing that every node on the relevant $M \cap \omega_1$-level of S_β is generic over the chosen countable elementary submodel. Equivalently, no uncountable antichain of S_β is added at the countable-support limit. Hence S_β remains Suslin in $W'^{\mathbb{P}_\lambda}$.

□

We end with a straightforward lemma which is used later in coding arguments.

Lemma 2.20. *Let T be a Suslin tree and let $\mathbb{A}_D(X)$ be the almost disjoint coding which codes a subset X of ω_1 into a real with the help of an almost disjoint family of reals D of size $\aleph_1$. Then*

$$\mathbb{A}_D(X) \Vdash T \text{ is Suslin}$$

holds.

Proof. Let $G \subseteq \mathbb{A}_D(X)$ be V-generic. Since $\mathbb{A}_D(X)$ has the Knaster property, the product $\mathbb{A}_D(X) \times T$ is ccc. Hence T remains Suslin in $V[G]$. □

3 Coding machinery

We continue with the construction of the appropriate notions of forcing which we want to use in our proof. The goal is to first define coding forcings Code(w) for reals w, which will force that a certain Σ^1_3-formula $\Phi(w)$ becomes true in the resulting generic extension. The coding method is slightly different than the one in [9], [13] or [12] leading to a slicker presentation. Recall that when we talk about the $\vec{S}$-sequence we mean all the members with index $\geqslant 0$, and leave S_{-1} out.

We use throughout this section the identification fixed in the construction of W: each tree S_α from $\vec{S} \cup \{S_{-1}\}$ is identified, via the fixed canonical bijection between $[\omega_1]^\omega$ and ω_1, with a subset of ω_1. Thus expressions such as $\gamma \in S_\alpha$ or $S_\alpha(\gamma) = 1$ refer to this fixed coding of the tree as a subset of ω_1.

We will work over the model W. Fix reals x, y and a natural number m, and let $w = w_{x,y,m} \in 2^\omega$ be the fixed recursive real coding the triple (x, y, m). We now define the coding forcing

$$\text{Code}(x, y, m),$$

equivalently Code(w). The definition proceeds in two steps. First we force with S_{-1}; the resulting generic branch determines the part of the sequence $\vec{S}$ which is used for the coding. From this branch and the real w we then define a set $Y = Y_{x,y,m} \subseteq \omega_1$, and the second step codes Y into one real by the fixed almost disjoint coding forcing. If z_0, z_1 are reals, then

$$\text{Code}(z_0, z_1)$$

will mean the same construction applied to the fixed recursive real coding the ordered pair (z_0, z_1); thus the two-argument notation introduces no new forcing notion. In what follows we often write (x, y, m) for the real w coding the triple whenever this improves readability.

Before defining the set Y, we isolate the terminology for the branch added by the first step.

Definition 3.1 (Coding area). *Let $H_{-1} \subseteq S_{-1}$ be generic and let*

$$b = \bigcup H_{-1} \subseteq S_{-1}$$

be the canonical cofinal branch induced by that filter. Identifying S_{-1} with a subset of ω_1 by the fixed coding of trees, we also identify b with the corresponding subset of ω_1. This subset b is called the coding area *determined by H_{-1}.*

For a fixed real w, the coding area b designates the ω-blocks of $\vec{S}$ used to record the bits of w: for $\gamma \in b$, the pair of trees

$$S_{\omega\cdot\gamma+2n}, \qquad S_{\omega\cdot\gamma+2n+1}$$

will record the n-th bit of w. Distinct uses of the coding construction use distinct generic branches through S_{-1}; hence their coding areas have countable intersections.

We now carry out the first step of the definition of Code(x, y, m). Force with S_{-1}, let $H_{-1} \subseteq S_{-1}$ be the generic filter, and let $b = \bigcup H_{-1}$ be the associated coding area. We work in $W[b]$ and define the crucial set $Y \subseteq \omega_1$ which will be used in the second step.

Let

$$\begin{aligned} A := &\{\omega\gamma + 2n \mid \gamma \in b, n \notin (x, y, m)\} \cup \\ &\{\omega\gamma + 2n + 1 \mid \gamma \in b, n \in (x, y, m)\}. \end{aligned}$$

Before choosing the auxiliary set X, we fix the coding convention used in this paragraph. Let $\pi_k : (\omega_1)^k \to \omega_1$, for $1 \leqslant k < \omega$, be the $<_L$-least system of bijections. We use these maps, and their finite iterates, to code finite tuples of subsets of ω_1 and ω_1-sequences of subsets of ω_1 by a single subset of ω_1. Thus, for instance, an ω_1-sequence $\langle X_\xi : \xi < \omega_1 \rangle$ is coded by

$$\{\pi_2(\xi, \eta) : \xi < \omega_1 \text{ and } \eta \in X_\xi\}.$$

The coding and decoding relation is fixed once and for all and is uniformly $\Sigma_1(\{\omega_1\})$-definable from the canonical L-choices. In the sequel, when we say that a subset of ω_1 *codes* several subsets or sequences of subsets of ω_1, we mean this fixed coding.

Let $X \subset \omega_1$ be chosen, according to this convention, so that it codes the following objects:

- the branch $b = \bigcup H_{-1} \subset S_{-1}$ induced by the generic filter $H_{-1} \subseteq S_{-1}$;
- the set $A \subset \omega_1$;
- the sequence of branches through the trees S_ξ, for those indices ξ of the form $\omega\gamma + 2n$ or $\omega\gamma + 2n + 1$ which belong to A, where each branch is the one induced by the corresponding generic filter for the tree forcing S_ξ;
- the sequence of clubs through the relevant members of $\vec{R}$ which are needed to define the trees from $\vec{S} \cup \{S_{-1}\}$ by the $\Sigma_1(\{\omega_1\})$-formula from the previous section, where each club is the union of the closed bounded approximations appearing in the corresponding club-shooting generic filter.

Let us name explicitly the formulas which express this decoding. If Z is either a real or a subset of ω_1, and if $\xi < \omega_1$, write

$$\mathrm{Br}(Z, \xi) \iff L[Z] \models \text{“}S_\xi \text{ has an } \omega_1\text{-branch.”}$$

For a real w and an ordinal $\gamma < \omega_1$, define

$$\text{BlockCode}(Z, \gamma, w) \iff \forall n \in \omega \Big((n \in w \leftrightarrow \text{Br}(Z, \omega \cdot \gamma + 2n + 1)) \\ \wedge (n \notin w \leftrightarrow \text{Br}(Z, \omega \cdot \gamma + 2n)) \Big).$$

When w is the recursive real coding the triple (x, y, m), we also write

$$\text{BlockCode}(Z, \gamma, x, y, m).$$

Thus X has been chosen so that

$$\forall \gamma \in b \ \text{BlockCode}(X, \gamma, x, y, m).$$

Indeed if $n \notin (x, y, m)$ then we added a cofinal branch through $S_{\omega \cdot \gamma + 2n}$. If on the other hand $S_{\omega \cdot \gamma + 2n}$ does not have an ω_1-branch in $L[X]$, then we must have added an ω_1-branch through $S_{\omega \cdot \gamma + 2n+1}$, since we always add an ω_1-branch through exactly one of the two trees $S_{\omega \cdot \gamma + 2n}$ and $S_{\omega \cdot \gamma + 2n+1}$; adding branches through the selected trees does not affect the fact that the remaining trees stay Suslin in $L[X]$, because $\vec{S}$ forms an independent family of Suslin trees.

We now use the localization trick from David's construction of a very absolute Π^1_2 real singleton [4]. We rewrite the information of X as a subset $Y \subset \omega_1$ using the following line of reasoning.

No particular model M is singled out here. The following argument applies to any transitive, $\aleph_1$-sized model M of ZF^- which contains X and the relevant parameters. Such an M first defines $\vec{S}$ correctly and then decodes from X the assertion $\text{BlockCode}(X, \gamma, x, y, m)$ at each ω-block of $\vec{S}$ starting at a point $\gamma \in b$. Consequently, if we code the model $(M, \in)$ which contains X as a set $X_M \subset \omega_1$, then for any ordinal β with $\omega_1 < \beta$, $|\beta| = \aleph_1$, and $L_\beta[X_M] \models \mathsf{ZF}^-$, we have

$$L_\beta[X_M] \models \exists N \Big(N \text{ is the transitive model decoded from } X_M \text{ and } \forall \gamma \in b_N \\ \text{BlockCode}^N(X_N, \gamma, x, y, m) \Big),$$

where X_N and b_N denote the objects decoded inside N according to the fixed convention, and BlockCode^N is the formula BlockCode interpreted inside N. In particular, choosing such a β ensures that X_M is available as an element of $L_\beta[X_M]$. We can then fix a club $C \subset \omega_1$ and a sequence $(M_\alpha : \alpha \in C)$ of countable elementary submodels of $L_\beta[X_M]$ such that

$$\forall \alpha \in C \, (M_\alpha \prec L_\beta[X_M] \wedge M_\alpha \cap \omega_1 = \alpha).$$

Now let the set $Y \subset \omega_1$ code the pair (C, X_M) such that the odd entries of Y code X_M and, if $E(Y)$ denotes the set of even entries of Y and $\langle c_\alpha : \alpha < \omega_1 \rangle$ is the increasing enumeration of C, then

1. $E(Y) \cap \omega$ codes a well-ordering of type c_0.
2. $E(Y) \cap [\omega, c_0) = \varnothing$.
3. For all $\alpha < \omega_1$, $E(Y) \cap [c_\alpha, c_\alpha + \omega)$ codes a well-ordering of type $c_{\alpha+1}$.
4. For all $\alpha < \omega_1$, $E(Y) \cap [c_\alpha + \omega, c_{\alpha+1}) = \varnothing$.

This gives the following local formula. For a subset $Y \subseteq \omega_1$ and a real w, let

$$\begin{aligned}\mathrm{LocalCode}(Y, w) \quad \iff \quad \forall M \Big(& M \text{ is countable and transitive, } M \models \mathsf{ZF}^-, \aleph_1^M \text{ exists,}\\ & \omega_1^M = (\omega_1^L)^M,\ Y \cap \omega_1^M \in M\\ & \to L[Y \cap \omega_1^M]^M \models \exists N\, \Theta(N, w)\Big),\end{aligned}$$

where $\Theta(N, w)$ abbreviates the first-order assertion that N is a transitive ZF^--model of size $\aleph_1^M$ which decodes out of $(Y \cap \omega_1^M)$ objects X_N and b_N according to the fixed convention and satisfies

$$\forall \gamma \in b_N \ \ \mathrm{BlockCode}^N(X_N, \gamma, w).$$

The construction above shows that $\mathrm{LocalCode}(Y, x, y, m)$ holds.

At this point the promised two-step forcing is defined. Let $\dot{Y}_{x,y,m}$ be the canonical S_{-1}-name for the set Y obtained by the construction above from the S_{-1}-generic branch and the real $w = w_{x,y,m}$. We set

$$\mathrm{Code}(x, y, m) = \mathrm{Code}(w) := S_{-1} * \dot{\mathbb{A}}_D(\dot{Y}_{x,y,m}),$$

where $\mathbb{A}_D$ is the almost disjoint coding forcing relative to the previously fixed almost disjoint family $D \in L$ (see the paragraph after Definition 2.5). Equivalently, after the first step has produced the coding area b and the associated set $Y = Y_{x,y,m}$, the second step is $\mathbb{A}_D(Y)$ and adds one real r coding Y. This second forcing only depends on the subset of ω_1 being coded, hence $\mathbb{A}_D(Y)$ is independent of the surrounding universe in which it is defined, as long as that universe has the right ω_1 and contains Y.

The effect of $\mathrm{Code}(x, y, m)$ is that it generically adds a real r satisfying the following formula:

$$\begin{aligned}\mathrm{RealCode}(r, w) \quad \iff \quad \forall M \Big(& M \text{ is countable and transitive, } M \models \mathsf{ZF}^-, \aleph_1^M \text{ exists,}\\ & \omega_1^M = (\omega_1^L)^M,\ r, w \in M\\ & \to L[r]^M \models \exists N\, \Theta(N, w)\Big).\end{aligned}$$

Again, when w is the recursive code of (x, y, m), we write $\mathrm{RealCode}(r, x, y, m)$.

Indeed, if r and M are as above, then M and $L[r]^M$ compute the almost disjoint family D up to the real indexed by ω_1^M correctly, as discussed below Definition 2.5. Consequently, $L[r]^M$ contains $Y \cap \omega_1^M$. Since

LocalCode(Y, x, y, m) holds, the model $L[r]^M$ sees a transitive ZF^--model N of size $\aleph_1^M$ satisfying $\Theta(N, (x, y, m))$, as required.

The formula RealCode(r, w) is Π^1_2 in the parameters r and w. We say in this situation that the real w, or equivalently the triple it codes, is written into $\vec{S}$. Thus, for any real $w = (x, y, m)$, the forcing Code(x, y, m) adds a real r such that RealCode(r, x, y, m) holds in the resulting extension.

The formula RealCode is also upward stable in the following sense. Although it is defined by quantifying over countable transitive models, if RealCode(r, w) holds and M is any transitive model of "ZF^-+ $\aleph_1$ exists and $\aleph_1 = \aleph_1^L$" with $r, w \in M$, then M satisfies the corresponding instance of the implication in the definition of RealCode. In other words, if RealCode(r, w) holds, then it also holds for uncountable, transitive models M. Otherwise, by Löwenheim–Skolem, a countable elementary submodel of such an M would collapse to a countable transitive counterexample.

Consequently, if RealCode(r, x, y, m) holds, then r carries enough information that, for every γ in the branch decoded from r, the universe $L[r]$ sees the branch pattern

$$n \in (x, y, m) \Rightarrow L[r] \models \text{``}S_{\omega\gamma+2n+1} \text{ has an } \omega_1\text{-branch''},$$
$$n \notin (x, y, m) \Rightarrow L[r] \models \text{``}S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch''}.$$

This follows by upward Σ_1-absoluteness and by the fact that the transitive models appearing in Θ compute $\vec{S}$ correctly.

4 Allowable forcings

Next we define the set of forcings which we will use in our proof. We aim to iterate the coding forcings we defined in the last section. The coding forcings should take care of two tasks simultaneously: on the one hand it should work towards Π^1_3-uniformization, on the other hand it should produce a Δ^1_3-definable well-order of the reals. These two tasks are reflected in the way we define allowable, which uses two cases.

The second case is used to produce a Δ^1_3-definable well-order of the reals. We first fix the following convention on names. Let $\mathbb{P}_\beta$ be an initial segment of an iteration. A set $A \subseteq \beta$, $A \in W$, is called *admissible for* $\mathbb{P}_\beta$ if the restriction of the iteration to the coordinates in A is recursively defined and yields a forcing $\mathbb{P}_A$ which is a complete subforcing of $\mathbb{P}_\beta$. In this case every $\mathbb{P}_A$-name σ has a canonical lift $\sigma^{\uparrow\beta}$ to a $\mathbb{P}_\beta$-name, and every $\mathbb{P}_\beta$-generic filter G_β induces a $\mathbb{P}_A$-generic filter $G_A = G_\beta \cap \mathbb{P}_A$.

If $\dot{x}$ is a $\mathbb{P}_\beta$-name for a real, a *localized presentation* of $\dot{x}$ is a pair (A, σ) such that $A \subseteq \beta$ is admissible for $\mathbb{P}_\beta$, σ is a $\mathbb{P}_A$-name for a real, and

$$1_{\mathbb{P}_\beta} \Vdash \dot{x} = \sigma^{\uparrow\beta}.$$

We order localized presentations by the canonical $<_L$-well-order, using the fixed $<_L$-coding of pairs. The canonical localized presentation of $\dot{x}$ is the $<_L$-least localized presentation of $\dot{x}$. This definition includes the case $A = \beta$, so such a presentation always exists.

We now impose the bookkeeping convention used in the rest of the paper. Bookkeeping functions are elements of L. Their values are not arbitrary objects of W, but L-codes which are decoded relative to the current initial iteration. Thus, when the current initial segment is $\mathbb{P}_\beta$, the value $F(\beta)$ is read as a tagged finite tuple of canonical $\mathbb{P}_\beta$-names, whenever the code has the appropriate form. Since the preparation from L to W is ω-distributive, no new countable sequences of ordinals are added; in particular the hereditarily countable codes for the real names which can occur in the countable-support constructions are already in L. We shall usually suppress the decoding map and write, for example, $F(\beta) = (\dot{x}, \dot{y}, \dot{m})$ or $F(\beta) = (\dot{z}_0, \dot{z}_1)$. This always means that the L-code $F(\beta)$, decoded relative to the already constructed $\mathbb{P}_\beta$, yields the displayed tuple of names.

Now suppose that the decoded value of $F(\beta)$ is $(\dot{z}_0, \dot{z}_1)$, where $\dot{z}_0$ and $\dot{z}_1$ are $\mathbb{P}_\beta$-names for reals. Let (A_i, σ_i) be the canonical localized presentation of $\dot{z}_i$, for $i < 2$. If $G_\beta \subseteq \mathbb{P}_\beta$ is generic, set $z_i = \dot{z}_i^{G_\beta}$; equivalently, $z_i = \sigma_i^{G_{A_i}}$, where G_{A_i} is the induced generic for $\mathbb{P}_{A_i}$. We define the auxiliary well-order used by the construction by declaring

$$z_0 < z_1 \quad \text{if and only if} \quad (A_0, \sigma_0) <_L (A_1, \sigma_1).$$

The allowable iteration codes the pair in the order determined by these canonical localized presentations.

We now define the base class of iterations used in the construction.

Definition 4.1 (0-allowable iterations)**.** *Let W be our ground model, let $\alpha < \omega_1$, and let $F \in L$, $F : \alpha \to H(\omega_2)^L$, be a bookkeeping function in the sense fixed above. A countable-support iteration*

$$\mathbb{P} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta \leqslant \alpha,\ \beta < \alpha \rangle$$

in W is called 0-allowable relative to F *if, for every $\beta < \alpha$, the β-th iterand is determined by the decoded value of $F(\beta)$ relative to the already constructed initial segment $\mathbb{P}_\beta$, as follows.*

1. *If the decoded bookkeeping value is*

$$(\dot{x}, \dot{y}, \dot{m}),$$

where $\dot{x}, \dot{y}$ are $\mathbb{P}_\beta$-names for reals and $\dot{m}$ is a $\mathbb{P}_\beta$-name for a natural number, then $\dot{\mathbb{Q}}_\beta$ is the canonical $\mathbb{P}_\beta$-name for

$$\mathrm{Code}(\dot{x}, \dot{y}, \dot{m}).$$

Equivalently, for every $\mathbb{P}_\beta$-generic filter G_β over W, if $x = \dot{x}^{G_\beta}$, $y = \dot{y}^{G_\beta}$ and $m = \dot{m}^{G_\beta}$, then

$$(\dot{\mathbb{Q}}_\beta)^{G_\beta} = \mathrm{Code}(x, y, m) = S_{-1} * \dot{\mathbb{A}}(\dot{Y}_{x,y,m}).$$

2. *If the decoded bookkeeping value is*

$$(\dot{z}_0, \dot{z}_1),$$

where $\dot{z}_0, \dot{z}_1$ are $\mathbb{P}_\beta$-names for reals, let (A_i, σ_i) be the canonical localized presentation of $\dot{z}_i$, for $i < 2$. If

$$(A_0, \sigma_0) <_L (A_1, \sigma_1),$$

then $\dot{\mathbb{Q}}_\beta$ is the canonical $\mathbb{P}_\beta$-name for

$$\mathrm{Code}(\dot{z}_0, \dot{z}_1).$$

If instead $(A_1, \sigma_1) <_L (A_0, \sigma_0)$, then $\dot{\mathbb{Q}}_\beta$ is the canonical $\mathbb{P}_\beta$-name for

$$\mathrm{Code}(\dot{z}_1, \dot{z}_0).$$

Equivalently, in any $\mathbb{P}_\beta$-generic extension, if $z_i = \dot{z}_i^{G_\beta} = \sigma_i^{G_{A_i}}$, then the next forcing is $\mathrm{Code}(z_0, z_1)$ *or* $\mathrm{Code}(z_1, z_0)$ *according to this fixed $<_L$-comparison.*

We shall usually say allowable *for* 0-allowable. *Thus allowability is a property of the given named countable-support iteration, witnessed by the L-bookkeeping function F; it does not involve choosing a particular generic filter during the construction.*

Informally speaking, the L-bookkeeping F, after decoding relative to the current initial segment, hands us at every step a real of the form (x, y, m) or a pair of reals (z_0, z_1), and the corresponding coding forcing uses the canonical generic branch through S_{-1} to determine the coding area in the sense of Definition 3.1. Thus each allowable coding block carries a specified almost disjoint subset $b \subseteq \omega_1$, namely its S_{-1}-generic branch, and the block writes bits only into the ω-blocks of $\vec{S}$ indexed by members of b. These coding areas are almost disjoint subsets of ω_1, and the definition requires that each such coding area is used at most once. This is the mechanism which prevents accidental coding of unwanted reals into $\vec{S}$.

Every allowable forcing $\mathbb{P}$ can be written over W as the countable-support iteration

$$\mathop{\ast}_{\beta<\delta}^{\mathrm{cs}} \big(S_{-1} * \mathbb{A}_D(\dot{Y}_\beta)\big).$$

The support convention in this section is the same as throughout the paper: all such iterations use countable support.

The next notion will become useful later.

Definition 4.2. *Let $\mathbb{P}$ be an allowable forcing over W relative to an L-bookkeeping $F : \delta_1 \to H(\omega_2)^L$, and let $\mathbb{Q}$ be an allowable forcing over W relative to the L-bookkeeping $H : \delta_2 \to H(\omega_2)^L$. Then we say $\mathbb{Q}$ is an allowable extension of $\mathbb{P}$, denoted by $\mathbb{Q} \rhd \mathbb{P}$, if $\delta_1 < \delta_2$ and $H \upharpoonright \delta_1 = F$.*

If $\mathbb{P} \in W$ is a forcing such that there are an $\alpha < \omega_1$ and an L-bookkeeping $F : \alpha \to H(\omega_2)^L$ such that $\mathbb{P}$ is allowable with respect to F, then we often just drop the F and simply say that $\mathbb{P} \in W$ is allowable.

We prove next some properties of allowable forcings.

Lemma 4.3. *1. If $\mathbb{P} = ((\mathbb{P}(\beta), \dot{\mathbb{P}}(\beta)) : \beta < \delta) \in W$ is allowable then for every $\beta < \delta$, $\mathbb{P}_\beta \Vdash |\dot{\mathbb{P}}(\beta)| = \aleph_1$.*

2. *Every allowable forcing over W is R-proper over L, in the following sense: the corresponding countable-support iteration over L obtained from the canonical names for its coding blocks is an R-proper forcing. Thus it preserves $\aleph_1$ and every countable set in an allowable extension can be covered by a countable set from L.*

3. *Every allowable forcing over W preserves $\aleph_2$ and CH. Furthermore, if $\mathbb{P} = (\mathbb{P}(\alpha) : \alpha < \omega_1) \in W$ is a countable-support ω_1-length iteration such that each initial segment of the iteration is allowable over W, then $W^{\mathbb{P}} \models \mathsf{CH}$.*

4. *Assume that $\mathbb{P} = (\mathbb{P}_\beta \mid \beta < \delta_0)$ is allowable relative to F^0 and $\mathbb{Q} = (\mathbb{Q}_\beta \mid \beta < \delta_1)$ is allowable relative to F^1. Then the literal product partial order $\mathbb{P} \times \mathbb{Q}$ has a canonical allowable iteration presentation: there is an allowable countable-support iteration $\mathbb{R}$ relative to an L-bookkeeping F and a canonical dense embedding*

$$\mathbb{P} \times \mathbb{Q} \longrightarrow \mathbb{R}.$$

Equivalently, $\mathbb{P} \times \mathbb{Q}$ is forcing equivalent to this allowable iteration.

Proof. The first assertion follows from the definition of the coding blocks. For the second assertion, note that each coding block is the two-step forcing $S_{-1} * \dot{\mathbb{A}}_D(\dot{Y})$. The tree forcing S_{-1} is locally ccc, hence R-proper for every stationary $R \subseteq \omega_1$, and the almost disjoint coding forcing is ccc, hence again R-proper. Therefore the corresponding countable-support iteration over L is an iteration of R-proper forcings and is R-proper. The preservation of ω_1 and the covering property for countable sets follow from Shelah's preservation theorem for countable-support iterations of S-proper forcings, as stated in Abraham [1, Theorem 2.10]. For the third item we note that the preservation of ω_2, and of CH in the present CH-preserving context, follows from the corresponding preservation theorem in Abraham [1, Theorem 2.12].

For the fourth assertion we again use the covering property from Abraham [1, Theorem 2.10]: every countable set of ordinals in an R-proper extension is contained in a countable set of ordinals from the ground model. So in particular the two-step iteration $\mathbb{P} * \check{\mathbb{Q}}$ can be densely embedded into a countable-support iteration from the ground model. What is left is to find an L-bookkeeping F which witnesses allowability. The bookkeeping is the concatenation of F^0 with the canonical translation of F^1. More precisely, set $F(\beta) = F^0(\beta)$ for $\beta < \delta_0$. If $\xi < \delta_1$ and a $\mathbb{Q}_\xi$-name τ occurs in $F^1(\xi)$, replace τ by its canonical lift $\tau^{\uparrow\mathbb{P}}$ to a $\mathbb{P} * \check{\mathbb{Q}}_\xi$-name. The lift is defined recursively by

$$\tau^{\uparrow\mathbb{P}} = \{(\sigma^{\uparrow\mathbb{P}}, (1_\mathbb{P}, \check{q})) : (\sigma, q) \in \tau\}.$$

Equivalently, under the dense identification of $\mathbb{P} * \check{\mathbb{Q}}_\xi$ with $\mathbb{P} \times \mathbb{Q}_\xi$, the condition $(1_\mathbb{P}, \check{q})$ is represented by $(1_\mathbb{P}, q)$. Thus, whenever $G \subseteq \mathbb{P}$ and $H_\xi \subseteq \mathbb{Q}_\xi$ are mutually generic,

$$(\tau^{\uparrow\mathbb{P}})^{G*H_\xi} = \tau^{H_\xi}.$$

This translation is applied to arbitrary $\mathbb{Q}_\xi$-names, not merely to check-names; check-names are only the special case $\tau = \check{a}$. We then set $F(\delta_0 + \xi)$ to be the corresponding L-code obtained from $F^1(\xi)$ by replacing all decoded names by these canonical lifts. Since the names involved are hereditarily countable and the lifting operation is fixed in L, the resulting bookkeeping still belongs to L.

The well-order of the reals is defined from canonical localized presentations, i.e. from the $<_L$-least pairs (A, σ) whose canonical lifts name the relevant real. Hence the comparison of two reals is independent of whether the reals are viewed in $W[\mathbb{P}]$ or in the larger extension $W[\mathbb{P} \times \mathbb{Q}]$: any smaller presentation available in the larger extension is already represented by the corresponding admissible subiteration and therefore is part of the same $<_L$-minimization. Thus the concatenated countable-support iteration obtained from F is allowable. The literal product partial order $\mathbb{P} \times \mathbb{Q}$ is not itself being declared allowable as a syntactic object; rather, it is canonically densely embedded into this allowable iteration presentation.

□

For the rest of this work, we use the following convention. Whenever a product of allowable forcings is called allowable, this means that the product is being identified with the canonical allowable iteration presentation obtained by concatenating the corresponding bookkeepings and lifting all names to the later block. Formally, all later applications use the canonical dense embedding

$$\mathbb{P} \times \mathbb{Q} \longrightarrow \mathbb{P} * \check{\mathbb{Q}} \longrightarrow \mathbb{R},$$

where $\mathbb{R}$ is the countable-support allowable iteration presentation just described. Names for the product are translated along this dense embedding.

Thus no assertion below depends on treating the literal product order as an iteration.

Let $\mathbb{P} = (\mathbb{P}(\beta) : \beta < \delta)$ be an allowable forcing with respect to some L-bookkeeping F, and let $G \subseteq \mathbb{P}$ be generic over W. We now define which reals are intentionally written into $\vec{S}$ by the iteration. For each $\beta < \delta$, let w_β be the real coded by the β-th block, defined as follows. If the decoded value of $F(\beta)$ is $(\dot{x}, \dot{y}, \dot{m})$, let (A_x, σ_x) and (A_y, σ_y) be the canonical localized presentations of $\dot{x}$ and $\dot{y}$. Set

$$x = \sigma_x^{G_{A_x}}, \qquad y = \sigma_y^{G_{A_y}}, \qquad m = \dot{m}^G,$$

and let w_β be the fixed recursive real coding the triple (x, y, m). If the decoded value of $F(\beta)$ is $(\dot{z}_0, \dot{z}_1)$, let (A_i, σ_i) be the canonical localized presentation of $\dot{z}_i$, for $i < 2$, and set $z_i = \sigma_i^{G_{A_i}}$. Let $(i, j) = (0, 1)$ if $(A_0, \sigma_0) <_L (A_1, \sigma_1)$, and let $(i, j) = (1, 0)$ otherwise. Then w_β is the fixed recursive real coding the ordered pair (z_i, z_j); this is exactly the order used in the second case of Definition 4.1. We put

$$\text{Coded}(\mathbb{P}, F, G) = \{w_\beta : \beta < \delta\}.$$

Thus the definition uses the canonical localized presentations fixed above; it does not depend on an arbitrary choice of names or on replacing names by check-names. When $\mathbb{P}$, F, and G are clear from context, we simply call this the set of reals coded by $\mathbb{P}$.

Next we show that iterations of 0-allowable forcings do not accidentally add new elements to the set of reals defined by the Σ^1_3-formula

$$\Phi(w) \iff \exists r \ \text{RealCode}(r, w).$$

Here RealCode is the Π^1_2 relation defined in the previous section. When w is the recursive code of a triple (x, y, m), we also write $\Phi(x, y, m)$ for $\Phi(w)$.

We first make explicit the branch-pulling argument which will be used in the proof. Write

$$W = L[G^0][\vec{c}][G^2],$$

where G^0 is the $\mathbb{Q}^0$-generic adding $\vec{S} \cup \{S_{-1}\}$, $\vec{c} = (c_i : i < \omega_1)$ is the $\mathbb{Q}^1$-generic sequence of branches through the trees S_i, $i < \omega_1$, and G^2 is the $\mathbb{Q}^2$-generic. If $i < \omega_1$, set

$$B_i = \omega_1 \backslash \{i\}, \qquad W_i = L[G^0][\vec{c} \upharpoonright B_i][G^2].$$

Thus $W = W_i[c_i]$.

For an allowable iteration $\mathbb{P}$ relative to F and a generic $G \subseteq \mathbb{P}$, let w_β and b_β denote, respectively, the real coded at stage β and the coding area, i.e.

the S_{-1}-generic branch, used at that stage. Define the set of $\vec{S}$-coordinates explicitly used by the interpreted coding blocks by

$$U(\mathbb{P}, F, G) = \bigcup_{\beta<\delta} U_\beta,$$

where

$$U_\beta = \{\omega \cdot \gamma + 2n : \gamma \in b_\beta \text{ and } n \notin w_\beta\} \cup \{\omega \cdot \gamma + 2n + 1 : \gamma \in b_\beta \text{ and } n \in w_\beta\}.$$

The bookkeeping itself will not contribute any additional branch-support parameter. By convention it belongs to L, and hence to every model W_i. The only coordinates which have to be avoided in the branch-pulling argument are therefore the coordinates explicitly used by the interpreted coding blocks, namely the elements of $U(\mathbb{P}, F, G)$.

Lemma 4.4 (Omitting an unused branch coordinate)**.** *Let* $\mathbb{P} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta \leqslant \delta, \beta < \delta \rangle$ *be allowable relative to* F*, let* $G \subseteq \mathbb{P}$ *be generic over* W*, and let* $i < \omega_1$ *satisfy*

$$i \notin U(\mathbb{P}, F, G).$$

Then the following hold.

1. $\mathbb{P} \in W_i$*, and* G *is* $\mathbb{P}$*-generic over* W_i*.*
2. $W[G] = W_i[G][c_i]$*.*
3. S_i *remains Suslin in* $W_i[G]$*.*
4. *The last extension* $W_i[G] \subseteq W_i[G][c_i]$ *is the quotient obtained by forcing with* S_i*, and it adds no reals.*

Proof. We prove the first two assertions by induction on the length of the allowable iteration. More precisely, for each $\xi \leqslant \delta$ we show that the initial segment $\mathbb{P}_\xi$ is an element of W_i, that $G_\xi = G \cap \mathbb{P}_\xi$ is $\mathbb{P}_\xi$-generic over W_i, and that

$$W[G_\xi] = W_i[G_\xi][c_i].$$

The assertion is trivial for $\xi = 0$.

Assume it has been proved for ξ and consider the next coding block. Since $F \in L \subseteq W_i$ and $\mathbb{P}_\xi \in W_i$, the decoded bookkeeping value at stage ξ is the same in W_i as in W. Hence the canonical localized presentations of the names appearing at this stage, and therefore the real w_ξ to be coded, are computed in $W_i[G_\xi]$ exactly as in $W[G_\xi]$. The first coordinate of the next iterand is the fixed tree forcing S_{-1}, which is in $W_i[G_\xi]$. Let $h_\xi \subseteq S_{-1}$ be the generic filter added at this coordinate, and let $b_\xi = \bigcup h_\xi$ be the corresponding coding area.

In $W_i[G_\xi][h_\xi]$ we can form the set U_ξ of selected $\vec{S}$-coordinates for this block. The assumption $i \notin U(\mathbb{P}, F, G)$ implies in particular that $i \notin U_\xi$.

Hence the set $Y_\xi \subseteq \omega_1$ used for the almost disjoint coding at this stage is definable from w_ξ, b_ξ, the fixed sequence $\vec{S}$, the $\mathbb{Q}^2$-generic clubs, and the branches c_j for $j \in U_\xi$; all these objects are already in $W_i[G_\xi][h_\xi]$. Thus

$$Y_\xi \in W_i[G_\xi][h_\xi],$$

and the almost disjoint coding forcing $\mathbb{A}_D(Y_\xi)$ is the same partial order whether computed in $W_i[G_\xi][h_\xi]$ or in $W[G_\xi][h_\xi]$. The generic real for this almost disjoint coding coordinate is part of $G_{\xi+1}$, so the successor step gives

$$\mathbb{P}_{\xi+1} \in W_i, \qquad G_{\xi+1} \text{ is } \mathbb{P}_{\xi+1}\text{-generic over } W_i, \qquad W[G_{\xi+1}] = W_i[G_{\xi+1}][c_i].$$

At a limit stage $\lambda \leqslant \delta$, the iteration is the countable-support limit of the earlier initial segments. Since $\lambda < \omega_1$, the support of every condition is countable, and the definition of the limit forcing is absolute between W_i and $W_i[c_i]$ once all earlier iterands have been identified. The induction hypotheses for $\xi < \lambda$ therefore imply that $\mathbb{P}_\lambda \in W_i$, that G_λ is generic over W_i, and that $W[G_\lambda] = W_i[G_\lambda][c_i]$. This completes the induction and proves (1) and (2).

For (3), apply Lemma 2.19 in the model $L[G^0][\vec{c} \upharpoonright B_i][G^2]$. The iteration $\mathbb{P}$ belongs to this model by (1). Each of its factors is either the fixed tree forcing S_{-1} or an almost disjoint coding forcing $\mathbb{A}_D(Y)$; the latter preserves Suslin trees by Lemma 2.20. Since $i \notin B_i$, Lemma 2.19 gives that S_i is still Suslin after forcing with $\mathbb{P}$, i.e. in $W_i[G]$.

Finally, the extension from $W_i[G]$ to $W_i[G][c_i]$ is obtained by the ground-model tree forcing S_i. The quotient has the same countably closed dense presentation as in Corollary 2.8, with the i-th branch coordinate placed last. This presentation is unaffected by the $\mathbb{Q}^2$-part, which is ω-distributive, and by the above induction the allowable iteration uses only c_i-free data. The standard fusion argument in the dense presentation decides every name for a natural number, and hence the quotient forcing is ω-distributive over $W_i[G]$. In particular no real is added by adjoining c_i.

Equivalently we could also use the fact that forcing with the Suslin tree S_i over $W_i[G]$ is ω-distributive and hence does not add reals to obtain (4). □

Lemma 4.5. *If $\mathbb{P} \in W$ is allowable relative to F, $\mathbb{P} = (\mathbb{P}_\beta : \beta < \delta)$, $G \subseteq \mathbb{P}$ is generic over W, and* $\text{Coded}(\mathbb{P}, F, G) = \{w_\beta : \beta < \delta\}$ *is the set of reals coded by $\mathbb{P}$, then in $W[G]$*

$$\{w \in 2^\omega : \Phi(w)\} = \{w_\beta : \beta < \delta\}.$$

Proof. First, every intentionally coded real satisfies Φ. Indeed, if $w = w_\beta$, then the β-th coding block adds an almost disjoint coding real r_β such that $\text{RealCode}(r_\beta, w_\beta)$ holds. Thus $\Phi(w_\beta)$ holds in $W[G]$.

Conversely, suppose toward a contradiction that $w \in 2^\omega$ satisfies $\Phi(w)$ but $w \notin \{w_\beta : \beta < \delta\}$. Fix a real r with $\text{RealCode}(r, w)$. By the definition of RealCode and the discussion following it, r decodes a coding area $b \subseteq \omega_1$ and, for every $\gamma \in b$ and every $n < \omega$,

$$n \in w \Rightarrow L[r] \models \text{“}S_{\omega\cdot\gamma+2n+1} \text{ has an } \omega_1\text{-branch”},$$

and

$$n \notin w \Rightarrow L[r] \models \text{“}S_{\omega\cdot\gamma+2n} \text{ has an } \omega_1\text{-branch”}.$$

We now choose one coordinate of the original $\mathbb{Q}^1$-generic branch sequence which is required by the code for w but is omitted by all intended coding blocks. Since $\delta < \omega_1$, there are only countably many intended coding areas b_β. Distinct branches through S_{-1} meet only countably. Choose $\eta < \omega_1$ above all intersections of distinct intended coding areas.

There are two cases. If $b = b_{\beta_0}$ for some $\beta_0 < \delta$, then $w \neq w_{\beta_0}$, so choose $n < \omega$ with $w(n) \neq w_{\beta_0}(n)$ and then choose $\alpha \in b_{\beta_0}$ with $\alpha > \eta$. If b is distinct from every b_β, choose $\alpha \in b$ above η and outside $\bigcup_{\beta<\delta}(b \cap b_\beta)$, and choose any $n < \omega$. In both cases let

$$i = \begin{cases} \omega\cdot\alpha + 2n + 1, & \text{if } n \in w, \\ \omega\cdot\alpha + 2n, & \text{if } n \notin w. \end{cases}$$

Then $i \notin U(\mathbb{P}, F, G)$: in the first case the intended block using the same coding area selects the opposite member of the pair because $w(n) \neq w_{\beta_0}(n)$, and all other intended coding areas are disjoint from α above η; in the second case no intended coding area contains α. Thus Lemma 4.4 applies to this i.

Let $N_i = W_i[G]$. By Lemma 4.4, S_i is Suslin in N_i and $W[G] = N_i[c_i]$ adds no reals over N_i. Hence the real r belongs already to N_i. On the other hand, by the choice of i, the code witnessed by r asserts

$$L[r] \models \text{“}S_i \text{ has an } \omega_1\text{-branch”}.$$

Since $r \in N_i$ and N_i is a transitive model containing the definition of $\vec{S}$, we have $L[r] \subseteq N_i$. Therefore N_i itself contains an ω_1-branch through S_i, contradicting the fact that S_i is Suslin in N_i.

This contradiction shows that no real outside $\{w_\beta : \beta < \delta\}$ satisfies Φ, and the lemma follows. □

5 α-allowability

We now define the derivative hierarchy of allowable forcings. The base case is the notion already introduced in Definition 4.1:

$$\text{0-allowable} \quad = \quad \text{allowable}.$$

Thus “allowable” without an ordinal prefix always means “0-allowable”. The hierarchy is defined by transfinite recursion with separate successor and limit clauses. At a successor level we use a neutral parameter ρ with $\rho = \alpha + 1$. This is an indexing convention, not a shift: the successor definition below is the definition of ρ-allowability itself, and the test in clause (a) ranges over $\zeta < \rho$. Thus, when $\rho = \omega + 1$, it also includes the limit level ω. The displayed rules do not define $\rho + 1$-allowability.

At a nonzero limit level λ there is no new bookkeeping recursion. We set

$$\mathcal{A}^W_\lambda = \bigcap_{\zeta<\lambda} \mathcal{A}^W_\zeta,$$

where $\mathcal{A}^W_\zeta$ denotes the class of ζ-allowable pairs $(\mathbb{P}, \dot{I})$ over W. Equivalently, $(\mathbb{P}, \dot{I})$ is λ-allowable iff it is ζ-allowable for every $\zeta < \lambda$. At the level of underlying forcings, this is exactly the intersection of the earlier classes of ζ-allowable forcings.

We also fix the following convention, which will be used throughout the definition. A member of the hierarchy is not a pair $(\mathbb{P}, I)$ belonging to W. Rather, it is a pair

$$(\mathbb{P}, \dot{I}) \in W,$$

where $\mathbb{P}$ is the iteration and $\dot{I}$ is a $\mathbb{P}$-name. If $G \subseteq \mathbb{P}$ is generic over W, we write

$$I^G = \dot{I}^G.$$

The interpreted set I^G belongs to $W[G]$, not to W, and it is the set of tentative values for the uniformizing functions. Thus all later statements of the form $(x, y, m, \xi) \in I$ are to be read as statements about $(x, y, m, \xi) \in \dot{I}^G$ in the relevant generic extension.

We shall also need a canonical support convention for names. Suppose $\mathbb{P}_\beta$ is an initial segment of an iteration and $\vec{\tau}$ is a finite tuple of $\mathbb{P}_\beta$-names. A set $A \subseteq \beta$, $A \in W$, is admissible for $\vec{\tau}$ if the subiteration on A is recursively defined, is a complete subforcing $\mathbb{P}_A$ of $\mathbb{P}_\beta$, and every member of $\vec{\tau}$ is the canonical lift of a $\mathbb{P}_A$-name. There may be many admissible supports, and there need not be a unique inclusion-minimal one. We therefore define

$$\mathrm{supp}_{\mathrm{AP}}(\vec{\tau}, \mathbb{P}_\beta) := \text{the } <_L \text{-least admissible support for } \vec{\tau}.$$

This is unique by definition, but it need not be an ordinal. The notation $\mathbb{P}_A$ refers to the complete subforcing indexed by the support set A.

Definition 5.1 (Limit and successor ρ-allowable iterations)**.** *Let ρ be a nonzero ordinal, and assume that ζ-allowability has been defined for every $\zeta < \rho$. If ρ is a limit ordinal, then $(\mathbb{P}, \dot{I})$ is* ρ-allowable *precisely when it belongs to the intersection $\mathcal{A}^W_\rho = \bigcap_{\zeta<\rho} \mathcal{A}^W_\zeta$ defined above. Thus no new bookkeeping recursion is performed at limit stages.*

Now suppose that ρ is a successor ordinal, say $\rho = \alpha + 1$. Let $F \in L$ be a bookkeeping function of countable domain $\delta < \omega_1$. We say that $(\mathbb{P}_\delta, \dot{I}_\delta)$ is

ρ-allowable relative to F *if it is obtained in W by the following recursion. Every object chosen below is a partial order, a name, or an ordinal belonging to W.*

At each stage $\beta \leqslant \delta$ we construct an initial segment $\mathbb{P}_\beta \in W$ and a $\mathbb{P}_\beta$-name $\dot{I}_\beta \in W$. We start with the trivial forcing and with $\dot{I}_0$ the canonical name for the empty set. At a limit stage $\eta \leqslant \delta$, $\mathbb{P}_\eta$ is the countable-support limit of $(\mathbb{P}_\xi : \xi < \eta)$, and $\dot{I}_\eta$ is the canonical $\mathbb{P}_\eta$-name forced to be

$$\bigcup_{\xi<\eta} \dot{I}_\xi^{\dot{G}_\xi},$$

where $\dot{G}_\xi$ is the canonical $\mathbb{P}_\xi$-generic filter induced by $\dot{G}_\eta$. This is a name in W; no actual generic filter is selected.

Assume next that $\beta < \delta$ and that $(\mathbb{P}_\beta, \dot{I}_\beta)$ has been constructed. The value $F(\beta)$ is first decoded, relative to $\mathbb{P}_\beta$, as explained before Definition 4.1. The next iterand $\dot{\mathbb{Q}}_\beta$ and the next name $\dot{I}_{\beta+1}$ are then chosen in W as the $<_L$-least $\mathbb{P}_\beta$-names satisfying the forcing-relation clauses described below. Equivalently, after forcing with $\mathbb{P}_\beta$, these names have the stated interpretation; the notation G_β is only semantic shorthand and is not part of the construction.

We assume that the decoded bookkeeping value $F(\beta)$ is a tuple of names $(\dot{x}, \dot{y}, \dot{m})$. Let

$$A = \mathrm{supp}_{\mathrm{AP}}((\dot{x}, \dot{y}, \dot{m}), \mathbb{P}_\beta),$$

and regard the corresponding $\mathbb{P}_A$-names as lifted to $\mathbb{P}_\beta$. Then we distinguish two cases. In both cases all quantifiers range over objects of W: ordinals $\zeta < \rho$, names, and relative ζ-allowable extensions $(\mathbb{R}, \dot{J})$ already defined at earlier levels of the recursion.

(a) *There are an ordinal $\zeta < \rho$ and a $\mathbb{P}_A$-name $\dot{y}_0$ for a real such that, after lifting $\dot{y}_0$ to $\mathbb{P}_\beta$,*

$$\mathbb{P}_\beta \Vdash (\dot{x}, \dot{y}_0) \in A_{\dot{m}},$$

and no relative ζ-allowable extension $(\mathbb{R}, \dot{J}) \trianglerighteq_\zeta (\mathbb{P}_\beta, \dot{I}_\beta)$ in W can force this pair out of the relevant section of A. More explicitly, after canonically lifting the names to $\mathbb{R}$,

$$\mathbb{P}_\beta \Vdash \text{“there is no condition in } \mathbb{R}/\dot{G}_\beta \text{ which forces } (\dot{x}^{\uparrow\mathbb{R}}, \dot{y}_0^{\uparrow\mathbb{R}}) \notin A_{\dot{m}^{\uparrow\mathbb{R}}}.\text{”}$$

If this happens, choose ζ least, and then choose the $<_L$-least localized name for such a $\dot{y}_0$. Then, the next forcing we use at stage β is a coding forcing $\mathrm{Code}(\dot{x}, \dot{z}, \dot{m})$, where $\dot{z}$ is supplied by the bookkeeping if possible and forced to be different from $\dot{y}_0$, and otherwise is chosen by the same $<_L$-least-name convention. Then we update $\dot{I}_{\beta+1}^{\dot{G}_{\beta+1}}$ such that it is forced to satisfy

$$\dot{I}_{\beta+1}^{\dot{G}_{\beta+1}} = \dot{I}_\beta^{\dot{G}_\beta} \cup \{(\dot{x}^{\dot{G}_\beta}, \dot{y}_0^{\dot{G}_\beta}, \dot{m}^{\dot{G}_\beta}, \zeta)\}.$$

Equivalently, in W we add to $\dot{I}_\beta$ the canonical name for the quadruple determined by the chosen names for $\dot{x}, \dot{y}_0, \dot{m}$ and by $\check{\zeta}$.

(b) *Suppose on the other hand that case (a) fails. Then the bookkeeping determines a pair of names, or if the bookkeeping entry is not of the required form the $<_L$-least default pair of names is used, and the next iterand is the corresponding coding block $\mathrm{Code}(\dot{x}, \dot{y}, \dot{m})$. In this case no new tentative uniformizing value is added:*

$$\dot{I}_{\beta+1} = \dot{I}_\beta$$

(up to the canonical lift from $\mathbb{P}_\beta$ to $\mathbb{P}_{\beta+1}$).

If the decoded bookkeeping value is instead a pair $(\dot{z}_0, \dot{z}_1)$ of names for reals, the next iterand is the well-order coding block from Definition 4.1. Namely, let (A_i, σ_i) be the canonical localized presentation of $\dot{z}_i$, for $i < 2$. We force with $\mathrm{Code}(\dot{z}_0, \dot{z}_1)$ or $\mathrm{Code}(\dot{z}_1, \dot{z}_0)$, according to whether $(A_0, \sigma_0) <_L (A_1, \sigma_1)$ or $(A_1, \sigma_1) <_L (A_0, \sigma_0)$. This clause does not change $\dot{I}$.

If $\mathbb{P} = \mathbb{P}_\delta$ and there is a $\mathbb{P}$-name $\dot{I} = \dot{I}_\delta$ such that $(\mathbb{P}, \dot{I})$ is ρ-allowable relative to some $F \in L$, then we simply say that $\mathbb{P}$ is ρ-allowable. When $G \subseteq \mathbb{P}$ is generic, the associated interpreted set of tentative values is $I^G = \dot{I}^G$.

We shall also use the following harmless convention for limit levels. If λ is a nonzero limit ordinal, a bookkeeping object $F \in L$ witnesses that $(\mathbb{P}, \dot{I})$ is λ-allowable if F codes an ordinary bookkeeping presentation of the underlying allowable iteration together with a sequence $\langle F_\zeta : \zeta < \lambda \rangle$ of L-bookkeeping witnesses such that F_ζ witnesses ζ-allowability of the same pair $(\mathbb{P}, \dot{I})$, for every $\zeta < \lambda$. This does not change the class $\mathcal{A}^W_\lambda$: witnesses are always required to be L-coded, and at a limit level we simply package the lower-level L-witnesses into one master L-code when such a package is needed.

Remark. The preceding definition is a recursion in the ground model W. Generic filters such as G_β are not chosen during the construction; they appear only to explain what the canonical names constructed in W denote after forcing. The informal role of I^G is the following: if $(x, y, m, \zeta) \in I^G$, then y is a tentative value for the uniformizing function $f_m(x)$ of rank ζ. The final choice will first minimize the rank and then use the fixed $<_L$-least-name convention. This explanatory point is not part of the formal recursion.

Nonemptiness at limit levels. With the successor clause now available, we record the promised interpretation of the limit clause. Let λ be a nonzero limit ordinal. The intersection defining λ-allowability is always nonempty: the trivial countable-support iteration together with the canonical empty auxiliary name belongs to every earlier level, and hence belongs to $\mathcal{A}^W_\lambda$. This observation should not be read as saying that limit allowability is only witnessed trivially. If an L-bookkeeping function F contains genuine coding tasks, then the limit clause simply asks that the resulting iteration survive all earlier tests. For instance, suppose that F lists check-names for reals from L, and that at least one listed pair is nontrivial. At each stage at which such an entry is used, the successor definition at every rank prescribes the corresponding coding block in W. In this example, the countable-support iteration obtained by following F is nontrivial, while the same underlying iteration belongs to every earlier class $\mathcal{A}^W_\zeta$, $\zeta < \lambda$. It therefore belongs to $\mathcal{A}^W_\lambda$. The same applies to any bookkeeping entries for which the successor clauses prescribe definite coding blocks at all earlier ranks. Thus the limit level contains the trivial iteration, but it also contains the nontrivial iterations generated by ordinary bookkeeping functions that list real coding tasks.

Definition 5.2 (Relative allowability and allowable tails)**.** *Let* $(\mathbb{P}, \dot{I}) \in W$ *be a pair consisting of a countable-support iteration and a* $\mathbb{P}$*-name of the auxiliary kind used above. We define, by induction on* ρ*, what it means for a pair* $(\mathbb{R}, \dot{J}) \in W$ *to be a* relative ρ-allowable extension *of* $(\mathbb{P}, \dot{I})$*, and write*

$$(\mathbb{R}, \dot{J}) \rhd_\rho (\mathbb{P}, \dot{I}).$$

This means that $\mathbb{R}$ *is obtained from* $\mathbb{P}$ *by appending a countable-support iteration of length* $< \omega_1$*, that the auxiliary names are continued from the initial name* $\dot{I}$*, and that the appended recursion satisfies the same clauses as the absolute* ρ*-allowability definition, with one modification only: the initial pair is* $(\mathbb{P}, \dot{I})$*, not the trivial forcing together with the empty auxiliary name. For* $\rho = 0$ *this uses Definition 4.1. For nonzero successor* ρ *it uses Definition 5.1, and at successor stages the alternatives (a) and (b) are evaluated over the current initial segment of the appended iteration. The quantifier over lower-rank extensions in clause (a) is understood in this same relative sense. For limit* ρ*, relative* ρ*-allowability means relative* ζ*-allowability for every* $\zeta < \rho$*.*

Now let $G \subseteq \mathbb{P}$ *be generic over* W*. A forcing* $\mathbb{T} \in W[G]$ *is called a* ρ-allowable tail over $W[G]$ above $(\mathbb{P}, \dot{I})^G$ *if there is a relative* ρ*-allowable extension* $(\mathbb{R}, \dot{J}) \rhd_\rho (\mathbb{P}, \dot{I})$ *in* W *such that, in* $W[G]$*,* $\mathbb{T}$ *is isomorphic to the quotient forcing* $\mathbb{R}/G$*.*

Lemma 5.3. *Work in* W*. If* $\mathbb{P}$ *is* β*-allowable over* W *and* $\alpha < \beta$*, then* $\mathbb{P}$ *is* α*-allowable over* W*. Thus the sequence of* α*-allowable forcings (over* W*) is decreasing with respect to the* $\subset$*-relation.*

Proof. It suffices to treat the case in which β is a successor ordinal and α is either 0 or a successor ordinal. If β is a limit, the conclusion is immediate from the intersection definition. If α is a nonzero limit, then, for underlying forcings, α-allowability just means membership in all lower classes, so the zero and successor cases below imply the limit case.

Assume then that β is a successor and that $(\mathbb{P}, \dot{I})$ is β-allowable relative to an L-bookkeeping function F. We define a new L-bookkeeping function F' so that the α-allowable construction guided by F' produces the same underlying iteration. We do not try to reproduce $\dot{I}$, since the lemma concerns only the forcing $\mathbb{P}$.

Suppose that, up to stage γ, the new construction has produced the same initial segment $\mathbb{P}_\gamma$ as the original one. If the original construction uses the well-order coding clause, set $F'(\gamma) = F(\gamma)$; this clause is rank-independent. If the original construction uses the uniformization clause (b), again set $F'(\gamma) = F(\gamma)$. Indeed, in this case clause (a) fails for all ranks $< \beta$, hence also for all ranks $< \alpha$, so the α-construction makes the same clause (b) choice.

It remains to consider a stage where the β-construction uses clause (a). Let $\xi < \beta$ be the least rank witnessing clause (a), and let the resulting iterand be

$$\mathrm{Code}(\dot{x}, \dot{z}, \dot{m}).$$

If $\xi < \alpha$, the same witness is available to the α-construction, and the same leastness convention gives the same rank, names, and coding block. Set $F'(\gamma) = F(\gamma)$.

If $\alpha \leqslant \xi < \beta$, then no witness of rank $< \alpha$ exists, by minimality of ξ. Thus the α-construction is not forced into clause (a). We now let $F'(\gamma)$ be the fixed L-code which, relative to $\mathbb{P}_\gamma$, decodes to the tuple $(\dot{x}, \dot{z}, \dot{m})$. For $\alpha = 0$, Definition 4.1 then prescribes the coding block $\mathrm{Code}(\dot{x}, \dot{z}, \dot{m})$. For successor α, clause (b) of Definition 5.1 prescribes the same block. Thus a stage at which β-allowability had the additional restriction of clause (a) is simulated, for α-allowability, by the ordinary coding alternative.

At limit stages both constructions take countable-support limits. Therefore the recursion produces the same underlying iteration. Since all bookkeeping entries are L-codes for canonical localized names, the modified bookkeeping F' belongs to L. Hence $\mathbb{P}$ is α-allowable over W. □

Lemma 5.4 (Allowability of tails)**.** *Let $(\mathbb{P}_\delta, \dot{I}_\delta)$ be ρ-allowable over W, witnessed by an L-bookkeeping function F, and let $(\mathbb{P}_\gamma, \dot{I}_\gamma)$, $\gamma \leqslant \delta$, be the canonical initial segments and auxiliary names produced by this construction. If $\beta \leqslant \gamma \leqslant \delta$, then*

$$(\mathbb{P}_\gamma, \dot{I}_\gamma) \rhd_\rho (\mathbb{P}_\beta, \dot{I}_\beta)$$

as a relative extension in W. Consequently, if $G_\beta \subseteq \mathbb{P}_\beta$ is generic over W, then the quotient forcing $\mathbb{P}_\gamma/G_\beta$ is a ρ-allowable tail over $W[G_\beta]$ above $(\mathbb{P}_\beta, \dot{I}_\beta)^{G_\beta}$. If $\xi < \rho$, then the same quotient is also a ξ-allowable tail. Finally, relative allowable extensions are transitive: if

$$(\mathbb{Q}, \dot{J}) \rhd_\xi (\mathbb{P}_\gamma, \dot{I}_\gamma) \quad \textit{and} \quad (\mathbb{P}_\gamma, \dot{I}_\gamma) \rhd_\xi (\mathbb{P}_\beta, \dot{I}_\beta),$$

then $(\mathbb{Q}, \dot{J}) \rhd_\xi (\mathbb{P}_\beta, \dot{I}_\beta)$.

Proof. Restrict the construction witnessing the ρ-allowability of $(\mathbb{P}_\delta, \dot{I}_\delta)$ to the interval from β to γ. The countable-support iteration on this interval is appended to the base pair $(\mathbb{P}_\beta, \dot{I}_\beta)$, and the auxiliary names are exactly the names already produced by the original recursion. Since the successor clauses in Definition 5.1 are local, depending only on the current initial segment, the current auxiliary name, the bookkeeping entry, and the forcing relation over that current initial segment, this restricted recursion witnesses

$$(\mathbb{P}_\gamma, \dot{I}_\gamma) \rhd_\rho (\mathbb{P}_\beta, \dot{I}_\beta).$$

Interpreting this W-coded relative extension by a $\mathbb{P}_\beta$-generic filter G_β gives precisely the usual quotient iteration $\mathbb{P}_\gamma/G_\beta$, so the quotient is a ρ-allowable tail by Definition 5.2.

If $\xi < \rho$, apply the proof of Lemma 5.3 to the appended recursion rather than to an absolute construction starting from the trivial forcing. That proof only changes the subsequent bookkeeping and does not use any special property of the initial pair. Hence the same tail has a relative ξ-allowable presentation, and its quotient over G_β is a ξ-allowable tail.

For transitivity, concatenate the two W-coded relative extension presentations. The second presentation starts from the terminal pair of the first, and countable-support concatenation again satisfies the same local successor and limit clauses. Thus the concatenation witnesses the displayed relative ξ-allowable extension from $(\mathbb{P}_\beta, \dot{I}_\beta)$ to $(\mathbb{Q}, \dot{J})$. □

Lemma 5.5. *Let $\mathbb{P}^1$ and $\mathbb{P}^2$ be α-allowable forcings over W. Then the product partial order $\mathbb{P}^1 \times \mathbb{P}^2$ has a canonical α-allowable iteration presentation. Equivalently, it is forcing equivalent, via a canonical dense embedding, to an α-allowable countable-support iteration. Moreover, if F_i witnesses the α-allowability of*

$$\mathbb{P}^i = (\mathbb{P}^i_\beta : \beta < \delta_i)$$

for $i = 1, 2$, then this allowable presentation is witnessed by an L-bookkeeping object $F \in L$ canonically definable from F_1 and F_2.

Proof. We prove the statement by induction on α. Throughout the proof the object constructed is the concatenated countable-support iteration presentation of the product, not the literal product order. The case $\alpha = 0$ is the product-presentation closure for allowable forcings proved above; if F_1 and F_2 witness the two zero-allowable presentations, the bookkeeping for the product is the usual concatenation of F_1 with a shifted copy of F_2, with all names in the second block canonically lifted to the corresponding product initial segments.

Assume that the statement has been proved for every $\zeta \leqslant \alpha$. We prove it for $\alpha + 1$. The L-bookkeeping function F' first runs F_1 and then runs a shifted copy of F_2. To avoid interval-restriction notation, let

$$\iota : \delta_2 \longrightarrow [\delta_1, \delta_1 + \delta_2)$$

be the order isomorphism $\iota(\beta) = \delta_1 + \beta$. We define F' by setting $F'(\beta) = F_1(\beta)$ for $\beta < \delta_1$, and by setting $F'(\iota(\beta))$ to be $F_2(\beta)$ with every $\mathbb{P}^2_\beta$-name replaced by its canonical lift to the product initial segment $\mathbb{P}^1 \times \mathbb{P}^2_\beta$. This definition is made at the level of L-codes and uses no generic filter.

The first δ_1 stages reproduce $\mathbb{P}^1$. We show by induction on $\beta \leqslant \delta_2$ that the next β stages reproduce the canonical iteration presentation of the product initial segment $\mathbb{P}^1 \times \mathbb{P}^2_\beta$. Limit stages are immediate from countable support. Suppose the assertion is known at β.

Consider a shifted successor stage $\delta_1 + \beta$. If the corresponding stage of $\mathbb{P}^2$ uses the well-order coding clause, then the product construction uses the same clause, because the comparison of canonical localized presentations is made in L and is unchanged by adding the dummy first block $\mathbb{P}^1$.

It remains to compare the uniformization alternatives. All formal statements below are statements of the forcing relation in W. There is, however, a semantic dependence on generics which should be made explicit. If the names in the bookkeeping entry are interpreted by two different generics, the resulting reals may be different, and the informal assertion that clause (a) or clause (b) holds for those interpreted reals may therefore depend on the particular generic filter. We do not claim generic-independence here. The point used in the proof is only compatibility with projection: whenever a product generic is written as $G^1 \times G^2_\beta$ for the product initial segment, the shifted construction is compared with the second construction interpreted in the induced $\mathbb{P}^2_\beta$-generic filter G^2_β.

Write the shifted bookkeeping value as $(\dot{x}, \dot{y}, \dot{m})$, lifted from the second block. The definition of $(\alpha + 1)$-allowability tests clause (a) against all ranks

$$\zeta < \alpha + 1,$$

not merely against $\zeta < \alpha$. This is exactly Definition 5.1 with $\rho = \alpha + 1$.

Assume first that the second construction is in clause (b) at stage β. Thus, for every $\zeta < \alpha + 1$, the relevant forcing-relation statement is witnessed in W by a ζ-allowable extension

$$(\mathbb{Q}^\zeta, \dot{J}^\zeta) \rhd (\mathbb{P}^2_\beta, \dot{I}^2_\beta)$$

whose quotient has a condition forcing the lifted candidate out of $A_{\dot{m}}$. Since $\zeta \leqslant \alpha$, the induction hypothesis gives that $\mathbb{P}^1 \times \mathbb{Q}^\zeta$ has a canonical ζ-allowable iteration presentation. Hence the product presentation also has, for every $\zeta < \alpha + 1$, a ζ-allowable extension whose quotient has a condition forcing the same candidate out of $A_{\dot{m}}$. Therefore clause (a) fails in the product presentation and clause (b) applies there as well.

Conversely, suppose the product presentation is in clause (b) at the shifted stage. The names $(\dot{x}, \dot{y}, \dot{m})$ have canonical support contained in the second block. By the support convention preceding Definition 5.1, any ζ-allowable iteration presentation of a product extension witnessing clause (b) may be restricted to the complete subiteration on this support, i.e. to an extension of $\mathbb{P}^2_\beta$. This restriction is again ζ-allowable by Lemma 5.3. Thus the same forcing-relation witnesses exist over $\mathbb{P}^2_\beta$, and the second construction is also in clause (b). Consequently clause (a) occurs in the product presentation exactly when it occurs in the second construction, and the $<_L$-least witnesses are the canonical lifts of the witnesses chosen in the second construction.

Therefore the shifted construction chooses exactly the lift of the next iterand chosen by F_2 at stage β, and it chooses the corresponding lift of the auxiliary name. This proves the successor step. Hence F' witnesses that the canonical iteration presentation of $\mathbb{P}^1 \times \mathbb{P}^2$ is $(\alpha + 1)$-allowable. The construction of F' is canonical from F_1 and F_2, and because the inputs are L-bookkeeping codes, the resulting bookkeeping again belongs to L.

Finally let λ be a nonzero limit ordinal and assume the product-closure statement, including the bookkeeping assertion, has been proved at every level $\zeta < \lambda$. If $\mathbb{P}^1$ and $\mathbb{P}^2$ are λ-allowable, then by the limit intersection convention they are ζ-allowable for every $\zeta < \lambda$. The induction hypothesis gives that the canonical iteration presentation of $\mathbb{P}^1 \times \mathbb{P}^2$ is ζ-allowable for every $\zeta < \lambda$, and hence

$$[\mathbb{P}^1 \times \mathbb{P}^2]_{\mathrm{it}} \in \bigcap_{\zeta<\lambda} \mathcal{A}^W_\zeta = \mathcal{A}^W_\lambda,$$

where $[\mathbb{P}^1 \times \mathbb{P}^2]_{\mathrm{it}}$ denotes the canonical concatenated iteration presentation. Thus the product has a λ-allowable iteration presentation.

It remains only to record the bookkeeping witness at the limit. Decode the limit witnesses F_i as master bookkeeping objects, say with lower-level components $\langle F_i^\zeta : \zeta < \lambda\rangle$, for $i = 1, 2$. By the induction hypothesis, for each $\zeta < \lambda$ there is a canonical L-bookkeeping object $F^\zeta \in L$, definable from F_1^ζ and F_2^ζ, which witnesses the ζ-allowability of the same product presentation. Together with the ordinary concatenation bookkeeping determining the underlying product iteration, these lower-level bookkeeping codes are packaged into a master L-bookkeeping object $F \in L$, canonically definable from F_1 and F_2, witnessing λ-allowability of the product presentation. □

The next preservation lemma is best read first as a name-level statement. The construction never chooses an interpreted real as a value during the recursion. At a stage of an α-allowable iteration at which clause (a) applies, it chooses canonical names for the relevant reals and then adds the canonical name for the corresponding quadruple to the name of the tentative values. Only after forcing with the completed iteration do these names become actual reals in the interpreted set of tentative values.

Lemma 5.6. *Let $(\mathbb{P}, \dot{I})$ be α-allowable over W with respect to F of length $\eta < \omega_1$. Let G be $\mathbb{P}$-generic over W, and put $I^G = \dot{I}^G$. If $(x, y, m, \xi) \in I^G$*

for some $\xi < \alpha$, then in $W[G]$, $(x, y) \in A_m$. Moreover, for every W-coded relative ξ-allowable extension $(\mathbb{Q}, \dot{J}) \rhd_\xi (\mathbb{P}, \dot{I})$,

$$\mathbb{Q}/G \Vdash (x, y) \in A_m.$$

Proof. We first prove the assertion with names. Suppose that $\beta < \eta$ is a stage at which clause (a) of Definition 5.1 is used. Let

$$\dot{x}_\beta, \quad \dot{y}_\beta, \quad \dot{m}_\beta$$

be the canonical $\mathbb{P}_\beta$-names selected at this stage, and let $\xi_\beta < \alpha$ be the rank selected by clause (a). Thus $\dot{I}_{\beta+1}$ is, in W, the canonical $\mathbb{P}_{\beta+1}$-name obtained from the lift of $\dot{I}_\beta$ by adjoining the canonical name for

$$(\dot{x}_\beta, \dot{y}_\beta, \dot{m}_\beta, \check{\xi}_\beta).$$

More precisely, after interpreting by any $\mathbb{P}_{\beta+1}$-generic filter, this added name is interpreted as

$$(\dot{x}_\beta^{G_\beta}, \dot{y}_\beta^{G_\beta}, \dot{m}_\beta^{G_\beta}, \xi_\beta),$$

where G_β is the induced $\mathbb{P}_\beta$-generic filter. Clause (a) gives the forcing-relation statement

$$\mathbb{P}_\beta \Vdash (\dot{x}_\beta, \dot{y}_\beta) \in A_{\dot{m}_\beta},$$

and, more importantly, the following condition-level preservation statement: for no relative ξ_β-allowable extension $(\mathbb{R}, \dot{J}) \rhd_{\xi_\beta} (\mathbb{P}_\beta, \dot{I}_\beta)$ does $\mathbb{P}_\beta$ force that there is a condition in the quotient $\mathbb{R}/\dot{G}_\beta$ forcing

$$(\dot{x}_\beta^{\uparrow\mathbb{R}}, \dot{y}_\beta^{\uparrow\mathbb{R}}) \notin A_{\dot{m}_\beta^{\uparrow\mathbb{R}}}.$$

All names here are the canonical lifts of the names chosen at stage β.

We claim that this name-level assertion persists through the rest of the iteration. Let γ be any ordinal with $\beta \leqslant \gamma \leqslant \eta$. Since the whole iteration is α-allowable and $\xi_\beta < \alpha$, Lemma 5.4 gives a W-coded relative ξ_β-allowable extension presentation of the segment from $(\mathbb{P}_\beta, \dot{I}_\beta)$ to $(\mathbb{P}_\gamma, \dot{I}_\gamma)$. Equivalently, after interpreting by any $\mathbb{P}_\beta$-generic filter, the quotient $\mathbb{P}_\gamma/G_\beta$ is a ξ_β-allowable tail in the precise sense of Definition 5.2. If some condition in $\mathbb{P}_\gamma$ forced

$$(\dot{x}_\beta^{\uparrow\mathbb{P}_\gamma}, \dot{y}_\beta^{\uparrow\mathbb{P}_\gamma}) \notin A_{\dot{m}_\beta^{\uparrow\mathbb{P}_\gamma}},$$

then, below its restriction to $\mathbb{P}_\beta$, its tail component would be a quotient condition witnessing exactly the kind of ξ_β-allowable extension forbidden by clause (a) at stage β. This contradicts the preservation statement just recalled. Hence

$$\mathbb{P}_\gamma \Vdash (\dot{x}_\beta^{\uparrow\mathbb{P}_\gamma}, \dot{y}_\beta^{\uparrow\mathbb{P}_\gamma}) \in A_{\dot{m}_\beta^{\uparrow\mathbb{P}_\gamma}}$$

for every $\gamma \in [\beta, \eta]$.

The same name-level argument also handles further lower-rank extensions. Here $\mathbb{Q}$ need not be a tail which appears inside the originally fixed iteration $\mathbb{P}$; this is the point of the second assertion of the lemma. Let $(\mathbb{Q}, \dot{J}) \trianglerighteq_{\xi_\beta} (\mathbb{P}, \dot{I})$ be any W-coded relative ξ_β-allowable continuation of the terminal pair $(\mathbb{P}, \dot{I})$. The segment from $(\mathbb{P}_\beta, \dot{I}_\beta)$ to $(\mathbb{P}, \dot{I})$ is a relative ξ_β-allowable extension by Lemma 5.4, and the continuation from $(\mathbb{P}, \dot{I})$ to $(\mathbb{Q}, \dot{J})$ is relative ξ_β-allowable by assumption. By the transitivity clause of Lemma 5.4, their concatenation is a single relative ξ_β-allowable extension of $(\mathbb{P}_\beta, \dot{I}_\beta)$. Therefore, if a condition in the quotient from $\mathbb{P}_\beta$ to $\mathbb{Q}$ forced the lifted pair out of $A_{\dot{m}_\beta}$, that condition would witness exactly what clause (a) at stage β ruled out. Hence no such quotient condition exists.

We now pass from names to interpreted reals. Fix $G \subseteq \mathbb{P}$ generic over W and suppose that $(x, y, m, \xi) \in I^G$. By the definition of the names $\dot{I}_\gamma$ as unions of the canonical stage-by-stage names, there is a least stage $\beta < \eta$ at which this tuple is added. Thus clause (a) was used at β, and the canonical names selected there satisfy

$$\dot{x}_\beta^{G_\beta} = x, \qquad \dot{y}_\beta^{G_\beta} = y, \qquad \dot{m}_\beta^{G_\beta} = m, \qquad \xi_\beta = \xi.$$

Applying the name-level conclusion with $\gamma = \eta$ and then interpreting by G gives

$$W[G] \models (x, y) \in A_m.$$

Finally, let $(\mathbb{Q}, \dot{J}) \trianglerighteq_{\xi} (\mathbb{P}, \dot{I})$ be a relative ξ-allowable extension. If some condition in $\mathbb{Q}/G$ forced $(x, y) \notin A_m$, then, using the displayed names from the stage β at which the tuple entered $\dot{I}$, the corresponding quotient condition over G_β would contradict the name-level preservation statement for the concatenated relative ξ-allowable extension from $\mathbb{P}_\beta$ to $\mathbb{Q}$. Hence no condition in $\mathbb{Q}/G$ forces the negation, and therefore

$$\mathbb{Q}/G \Vdash (x, y) \in A_m.$$

□

The fact that the set of α-allowable forcings forms a non-empty, decreasing sequence of sets of partial orders implies that the sequence has to stabilize. More precisely: let $\mathcal{C}^W$ be the set of canonical $<_L$-codes in W for pairs $(\mathbb{P}, \dot{I})$ of the kind used above: $\mathbb{P}$ is a countable-support iteration of length $< \omega_1$ whose iterands have size $\aleph_1$, and $\dot{I}$ is an auxiliary name of the prescribed form. We always replace isomorphic or forcing-equivalent presentations by their $<_L$-least code. Thus, for each ordinal ρ, the class of ρ-allowable pairs is represented by a genuine subset

$$\mathcal{A}_\rho^W \subseteq \mathcal{C}^W.$$

By Lemma 5.3 the sequence $\langle \mathcal{A}^W_\rho : \rho \in \mathrm{Ord}\rangle$ is decreasing, and at limit stages it is defined by intersection. Since all its terms are subsets of the single set $\mathcal{C}^W$, it cannot be strictly decreasing through all ordinals. More explicitly, let

$$\kappa = (2^{|\mathcal{C}^W|})^+$$

be computed in W. There are at most $2^{|\mathcal{C}^W|}$ subsets of $\mathcal{C}^W$. If the sequence did not eventually stabilize, then one could recursively choose an increasing sequence $\langle \rho_\xi : \xi < \kappa\rangle$ such that

$$\mathcal{A}^W_{\rho_{\xi+1}} \subsetneq \mathcal{A}^W_{\rho_\xi}$$

for every $\xi < \kappa$. The sets $\mathcal{A}^W_{\rho_\xi}$, $\xi < \kappa$, would then be pairwise distinct subsets of $\mathcal{C}^W$. This is impossible, because $\kappa > 2^{|\mathcal{C}^W|} = |\mathcal{P}(\mathcal{C}^W)|$. Hence there is an ordinal α such that

$$\mathcal{A}^W_\beta = \mathcal{A}^W_\alpha \qquad \text{for all } \beta \geqslant \alpha.$$

By the nonemptiness observation following Definition 5.1, the stabilized class is nonempty.

Definition 5.7. *Let α_0 be the least ordinal α such that*

$$\mathcal{A}^W_\beta = \mathcal{A}^W_\alpha \qquad \textit{for every } \beta \geqslant \alpha.$$

This least ordinal exists by the preceding paragraph and is unique because the ordinals are well ordered.

Set

$$\alpha^* = \alpha_0 + 1.$$

Since the hierarchy has stabilized at α_0, the α^*-allowable pairs have exactly the same canonical codes as the α_0-allowable pairs. We use this successor level in the final construction so that the successor rules of Definition 5.1 apply literally, even if the first stabilization point α_0 happens to be a limit ordinal.

The phrase ∞-allowable will be used only as shorthand for the stabilized construction just obtained. More explicitly, if $\delta < \omega_1$, then an iteration $\mathbb{P} = (\mathbb{P}_\beta : \beta < \delta)$ of length δ is ∞-allowable if it is produced by the successor recursion for α^*-allowability, using the stabilized class of canonical codes. Thus the informal phrase "there is some ordinal ζ for which clause (a) applies" means the following statement inside W: there are an ordinal $\zeta < \alpha^*$ and canonical W-names witnessing clause (a), and there is no canonical code in $\mathcal{C}^W$ for a ζ-allowable extension which forces the displayed candidate out of the relevant section of A_m. Since the hierarchy has stabilized at α_0, this bounded search through $\zeta < \alpha^*$ is equivalent to the earlier informal search through all ranks. If the search succeeds, the recursion uses the $<_L$-least witnessing code. If it fails, the recursion uses the $<_L$-least code prescribed by the alternative clause. In both cases the decision is made by the forcing relation over partial orders and names which belong to W.

Consequently, the definition of an ∞-allowable iteration is a ground-model recursion on canonical codes, names, ordinals, and forcing-relation statements. No interpreted real, no interpreted auxiliary set, and no generic filter is used to make a choice. After a generic interpretation, for fixed $m \in \omega$ and a real x there may be several potential y's with (x, y, m, ξ) in the interpreted auxiliary set. These are only tentative values. The final uniformizing value is obtained after the forcing construction by first minimizing the rank ξ and then applying the fixed $<_L$-least-name convention. Since the ranks are ordinals, this post-construction minimization stabilizes and does not require choosing a generic filter during the definition of the iteration.

6 Definition of the universe in which the Π^1_3 uniformization property holds

The notion of ∞-allowable will be used now to define the universe in which the Π^1_3-uniformization property holds. Fix in L a bookkeeping function

$$F : \omega_1 \longrightarrow H(\omega_1)^L$$

such that every relevant L-code appears unboundedly often.[1]

We define in W, by recursion on $\beta < \omega_1$, a countable-support iteration $\mathbb{P}_{\omega_1}$ together with a coherent sequence of names $\langle \dot{I}_\beta : \beta \leqslant \omega_1 \rangle$. No generic filter is chosen during this recursion. At limit stages we take the countable-support limit and the canonical name for the union of the earlier $\dot{I}_\xi$'s.

At a successor stage $\beta + 1$, assume that $\mathbb{P}_\beta$ and $\dot{I}_\beta$ have already been constructed. The next iterand and the next auxiliary name are chosen in W by the following cases.

Uniformization bookkeeping. Suppose that the L-code $F(\beta)$ codes a triple (η_1, η_2, m) with $\eta_1 \leqslant \beta$, and let $(\dot{x}, \dot{y})$ be the η_2-th nice $\mathbb{P}_{\eta_1}$-name for a pair of reals, lifted canonically to a $\mathbb{P}_\beta$-name. We apply the stabilized α^*-test to these names. This is the successor clause (a) from Definition 5.1 with the quantifier ranging over all ranks $\xi < \alpha^*$, where limit ranks are interpreted by the limit-intersection convention above. Formally, this is a test in W: one searches through canonical $<_L$-codes for the relevant ordinals, names, and allowable extensions, and all positive and negative alternatives are evaluated by the forcing relation in W.

[1] This requirement on F is only a bookkeeping convention used in the verification; it is not an additional clause in the definition of allowability.

(U1) If this stabilized clause (a) applies, we choose in W the least witnessing code, equivalently the least pair $(\xi, \dot{y}_0)$ under the fixed coding convention. Then choose the least appropriate name $\dot{z} \in W$ with

$$\mathbb{P}_\beta \Vdash \dot{z} \neq \dot{y}_0$$

which has not already been intentionally coded at an earlier stage. Set

$$\dot{\mathbb{Q}}_\beta = \mathrm{Code}(\dot{x}, \dot{z}, \check{m}),$$

and let $\dot{I}_{\beta+1}$ be the canonical $\mathbb{P}_{\beta+1}$-name for

$$(\dot{I}_\beta)^{\dot{G}_\beta} \cup \{(\dot{x}, \dot{y}_0, \check{m}, \check{\xi})\}.$$

Here $\dot{G}_\beta$ is the canonical name for the induced $\mathbb{P}_\beta$-generic filter; this displayed formula is only the semantic description of the canonical name added in W.

(U2) If the stabilized clause (a) does not apply, we choose in W the $<_L$-least canonical code for a relative α^*-allowable extension

$$(\mathbb{R}, \dot{J}) \rhd_{\alpha^*} (\mathbb{P}_\beta, \dot{I}_\beta)$$

whose interpreted quotient forces the current candidate out of A_m. This means that, after canonically lifting $\dot{x}$ and $\dot{y}$ to $\mathbb{R}$, the forcing relation in W gives

$$\mathbb{P}_\beta \Vdash \text{“there is a condition in } \mathbb{R}/\dot{G}_\beta \text{ which forces } (\dot{x}^{\uparrow\mathbb{R}}, \dot{y}^{\uparrow\mathbb{R}}) \notin A_{\check{m}}.\text{”}$$

Here $\dot{G}_\beta$ is the canonical name for the induced $\mathbb{P}_\beta$-generic filter. The quantifier ranges over the fixed set $\mathcal{C}^W$ of canonical codes, and the displayed assertion about the quotient is a forcing-relation statement in W. Let $\dot{\mathbb{Q}}_\beta$ be the canonical $\mathbb{P}_\beta$-name, constructed in W, for this quotient, and let $\dot{I}_{\beta+1}$ be the corresponding quotient lift of $\dot{J}$.

Well-order bookkeeping. If the L-code $F(\beta)$ instead decodes to a pair of names $(\dot{z}_0, \dot{z}_1)$ for reals, we use the well-order coding clause. Let (A_i, σ_i) be the canonical localized presentation of $\dot{z}_i$, for $i < 2$.

(W1) If $(A_0, \sigma_0) <_L (A_1, \sigma_1)$, set

$$\dot{\mathbb{Q}}_\beta = \mathrm{Code}(\dot{z}_0, \dot{z}_1).$$

(W2) If $(A_0, \sigma_0) <_L (A_1, \sigma_1)$ fails, set

$$\dot{\mathbb{Q}}_\beta = \mathrm{Code}(\dot{z}_1, \dot{z}_0).$$

In this pair-coding case $\dot{I}_{\beta+1}$ is the canonical lift of $\dot{I}_\beta$.

This ends the definition, in W, of the pair $(\mathbb{P}_{\omega_1}, \dot{I}_{\omega_1})$. The bookkeeping F can be recoded into a single function $F' \in L$ such that, for every $\delta < \omega_1$, the initial segment $(\mathbb{P}_\delta, \dot{I}_\delta)$ is α^*-allowable over W relative to $F' \upharpoonright \delta$.

Fact 6.1. *The just defined pair $(\mathbb{P}_{\omega_1}, \dot{I}_{\omega_1})$ belongs to W, and every initial segment is α^*-allowable over W relative to a fixed $F' \in L$.*

As a consequence, the f_m-values of rank $< \alpha^*$ named during the iteration will certainly belong to A_m in the final model by Lemma 5.6. We let G_{ω_1} be $\mathbb{P}_{\omega_1}$-generic over W, put $I_{\omega_1} = (\dot{I}_{\omega_1})^{G_{\omega_1}}$, and write $I = I_{\omega_1}$. What is left is to show that in $W[G_{\omega_1}]$, for every $m \in \omega$ and every real x such that $A_{m,x} \neq \emptyset$, we do have exactly one pair of reals $(x, y) \in A_m$ such that (x, y, m) is not coded into $\vec{S}$. The next lemma does exactly that, and is the main step in proving that the Π^1_3-uniformization property holds true in $W[G_{\omega_1}]$.

Lemma 6.2. *In $W[G_{\omega_1}]$, for every real x and every $m \in \omega$, exactly one of the following alternatives holds.*

1. *There are a real y and an ordinal $\xi < \alpha^*$ such that $(x, y, m, \xi) \in I_{\omega_1}$. In this case there is a unique real y_0 such that*

   $W[G_{\omega_1}] \models$ "$(x, y_0) \in A_m \wedge (x, y_0, m)$ *is not coded somewhere into* $\vec{S}$".

2. *For every real y and every ordinal $\xi < \alpha^*$, $(x, y, m, \xi) \notin I_{\omega_1}$. In this case*

   $W[G_{\omega_1}] \models$ "*The x-section of A_m is empty.*"

Proof. The two alternatives are mutually exclusive and exhaustive by their first sentences. We prove the conclusion attached to each alternative.

Assume first that the first alternative holds. Choose a real $y_0 \in W[G_{\omega_1}]$ and an ordinal $\xi_0 < \alpha^*$ such that $(x, y_0, m, \xi_0) \in I_{\omega_1}$ and whose canonical $\mathbb{P}_{\omega_1}$-name is $<$-minimal among all names giving such a witness. Let β be the least stage such that $(x, y_0, m, \xi_0) \in \dot{I}_{\beta+1}^{G_{\beta+1}} \backslash \dot{I}_\beta^{G_\beta}$.

Claim 1. $W[G_{\omega_1}] \models (x, y_0) \in A_m$.

Proof of the first Claim. This follows immediately from the lemma 5.6. □

Claim 2.

$W[G_{\omega_1}] \models$ "*y_0 is the unique real such that*
(x, y_0, m) is not coded somewhere in the $\vec{S}$-sequence."

Proof of the second Claim. We shall prove the second claim. First we show that (x, y_0, m) is not coded somewhere into the $\vec{S}$-sequence. It is clear that from stage β on, we will not code (x, y_0, m) into $\vec{S}$. So the only possibility that we coded up (x, y_0, m) is that there is a stage $\eta < \beta$ of our iteration $\mathbb{P}_{\omega_1}$ where we coded (x, y_0, m) into $\vec{S}$. At stage η, we can not be in case 2, as (x, y_0) and the fact that we are in case 1 at stage β, witness that we must be in case 1 at η. So we must be in case 1, but we add a different (x, y', m, ξ_0) to the interpreted name $(\dot{I}_\eta)^{G_\eta}$, but its <-least name must be <-less than the <-least name for $(x, y_0, m,)$ which is a contradiction to our assumption.

In order to see that it is the unique real of the form (x, y, m) which is not coded, it is sufficient to note that for every other $y \neq y_0$, (x, y, m) will be coded into $\vec{S}$ by the rule (1) of our definition.

Thus Claim 2 is proved, which also finishes the proof of the Lemma under the assumptions of the first case of our Lemma. □

We shall prove now that under the assumptions of the second case of our Lemma, its conclusion does hold, i.e. we need to show that if (x, m) is such that for every real y and every $\xi < \alpha^*$, $(x, y, m, \xi) \notin I_{\omega_1}$, then $W[G_{\omega_1}] \models$ "The x-section of A_m is empty".

But under these assumptions, whenever we are at a stage β at which the L-bookkeeping decodes to canonical names whose interpretations are the triple (x, y, m), then case 2 of the definition of $\mathbb{P}_{\omega_1}$ must apply. But for every such y, at stage β, we ensure with an α^*-allowable forcing that $W[G_{\beta+1}] \models (x, y) \notin A_m$. By upwards absoluteness of Σ^1_3-formulas we obtain in the end

$$W[G_{\omega_1}] \models \neg \exists y((x, y) \in A_m).$$

This finishes the proof of the Lemma.

□

Corollary 6.3. *In $W[G_{\omega_1}]$ the Π^1_3-uniformization property is true. More precisely, if A_m is the m-th relation in the fixed universal list of lightface Π^1_3 subsets of $(\omega^\omega)^2$, define*

$$G_m(x, y) \iff (x, y) \in A_m \wedge \neg \exists r \text{ RealCode}(r, x, y, m).$$

Then G_m is a Π^1_3 relation and, whenever the x-section of A_m is nonempty, there is exactly one y such that $G_m(x, y)$.

Proof. By Lemma 6.2, if the x-section of A_m is nonempty, the second alternative is impossible, and the first alternative gives a unique real y such that $(x, y) \in A_m$ and (x, y, m) is not coded into $\vec{S}$. By the definition of "coded into $\vec{S}$" from the coding section, this last assertion is exactly

$$\neg \exists r \text{ RealCode}(r, x, y, m).$$

Thus G_m uniformizes A_m.

It remains only to record the projective complexity. The formula $(x, y) \in A_m$ is Π^1_3 by the choice of the universal list. The relation $\mathrm{RealCode}(r, x, y, m)$ was shown in the coding section to be Π^1_2 in the displayed parameters. Hence

$$\exists r \ \mathrm{RealCode}(r, x, y, m)$$

is a Σ^1_3 statement, and its negation is Π^1_3. Therefore $G_m(x, y)$ is the conjunction of two Π^1_3 statements, hence is again Π^1_3. □

Finally, the Δ^1_3 well-order of the reals follows from the corresponding statement about names. For every pair of reals (z_0, z_1) in the final extension there are canonical names $\dot{z}_0, \dot{z}_1$ and some stage $\beta < \omega_1$ at which the bookkeeping F lists the code for this pair of names. In case (3) at that stage, the construction codes exactly one of the ordered pairs (z_0, z_1) and (z_1, z_0) into $\vec{S}$, according to the fixed comparison of their canonical localized presentations. Thus we define in $W[G_{\omega_1}]$:

$$z_0 < z_1 \Leftrightarrow \ (z_0, z_1) \text{ is coded into } \vec{S}$$

the latter being a $\Sigma^1_3(z_0, z_1)$-statement, which is what we want.

7 Higher-level analogues

The construction in this paper is presented only for the L-based third projective level. We expect that corresponding higher-level analogues can be obtained over canonical inner models with finitely many Woodin cardinals by combining the present argument with the machinery developed in [9]. Since those adaptations require additional technical choices which are not used in the proof above, they are not stated as part of the main theorem of the present paper.

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