Forcing $\mathbf{\Sigma}^1_1$-Separation on $\omega_1^{\omega_1}$

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Abstract

We prove that it is consistent that every two disjoint boldface $\mathbf{\Sigma}^1_1$ subsets of 2^{ω_1} can be separated by a boldface $\mathbf{\Delta}^1_1$ set. This is the first construction of a universe in which the $\mathbf{\Sigma}^1_1$-separation property holds at ω_1. The forcing starts from L and preserves CH and therefore also $\omega_1^{<\omega_1} = \omega_1$.

1 Introduction

Generalized descriptive set theory studies the spaces κ^κ and 2^κ for uncountable regular cardinals κ. Already for $\kappa = \omega_1$, the structure of the definable pointclasses is much less rigid than in the classical case. In particular, the relations between Borel sets, Borel* sets, $\mathbf{\Delta}^1_1$ sets and $\mathbf{\Sigma}^1_1$ sets become sensitive to forcing and to the ambient universe [33, 7, 25, 30, 32, 31, 8, 3, 2, 29]. One of the central regularity principles for analytic sets is separation. In classical descriptive set theory, any two disjoint analytic sets can be separated by a Borel set [28, 35]. In the generalized setting, the corresponding question becomes more delicate. The usual ZFC separation theorem gives Borel* separators for disjoint $\mathbf{\Sigma}^1_1$ sets [33, 7, 25], but this leaves open whether the separator can consistently be chosen at the $\mathbf{\Delta}^1_1$ level. The main result of this paper is a positive consistency theorem for this stronger separation principle. We force a model in which the full $\mathbf{\Sigma}^1_1$-separation property holds on 2^{ω_1}: whenever

$$A_0, A_1 \subseteq 2^{\omega_1}$$

are disjoint $\mathbf{\Sigma}^1_1$ sets, there is a $\mathbf{\Delta}^1_1$ set D such that

$$A_0 \subseteq D \quad \text{and} \quad D \cap A_1 = \emptyset.$$

Thus the theorem is asserting a global separation principle for all analytic separation problems appearing in the final model.

Theorem 1.1 (Main theorem)**.** *Assume $V = L$. There is a cardinal-preserving forcing extension satisfying*

$$2^\omega = \omega_1, \qquad 2^{\omega_1} = \omega_2,$$

in which every two disjoint $\mathbf{\Sigma}^1_1$ subsets of 2^{ω_1} are separated by a $\mathbf{\Delta}^1_1$ set.

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The final model lies in the usual small-base regime $\omega_1^{<\omega_1} = \omega_1$. In this setting the standard tree-projection presentation of $\mathbf{\Sigma}^1_1$ agrees with its $\Sigma_1(H(\omega_2))$ presentation, and the resulting pointclass has the expected closure and absoluteness properties used throughout the construction [30]. The theorem also has a natural interpretation in connection with one of the basic open problems about the generalized Borel hierarchy. In the classical case $\kappa = \omega$, the classes Borel, Borel* and $\mathbf{\Delta}^1_1$ coincide. At uncountable cardinals the situation is different. In the usual context $\kappa^{<\kappa} = \kappa$, one has

$$\text{Borel} \subseteq \mathbf{\Delta}^1_1 \subseteq \text{Borel}^* \subseteq \mathbf{\Sigma}^1_1.$$

The possible relation between Borel* and $\mathbf{\Delta}^1_1$ remains a central question. If a model of Borel* $= \mathbf{\Delta}^1_1$ were available, then the known Borel* separation theorem would immediately imply $\mathbf{\Sigma}^1_1$-separation by $\mathbf{\Delta}^1_1$ sets. From this perspective, Theorem 1.1 proves the consistency of the separation-theoretic consequence of the possible identity Borel* $= \mathbf{\Delta}^1_1$. It does not decide the global equality. Rather, it shows that one of the most visible consequences of such an equality can be forced directly: for analytic separation problems, Borel* separation can consistently be sharpened all the way down to $\mathbf{\Delta}^1_1$ separation. The construction should also be viewed as part of a broader forcing program around the implication pattern

$$\text{uniformization} \Rightarrow \text{reduction} \Rightarrow \text{separation}.$$

At the classical projective levels, this program includes forcing $\mathbf{\Sigma}^1_3$-separation, separating $\mathbf{\Pi}^1_3$-reduction from $\mathbf{\Pi}^1_3$-uniformization, forcing $\mathbf{\Pi}^1_n$-uniformization, and combining or separating uniformization, reduction and separation principles by forcing [12, 14, 10, 15, 18, 22, 13, 17, 23]. The present paper is the generalized 2^{ω_1} separation analogue of that program. The stationary-coding and Suslin-tree coding background also connects the present work with results on definability of the nonstationary ideal and with projective well-order and coding constructions [6, 11, 24, 20, 21, 19, 16]. We emphasize, however, that the argument in this paper is tailored to separation. It does not presently force $\mathbf{\Pi}^1_1$-reduction or $\mathbf{\Pi}^1_1$-uniformization on 2^{ω_1}, nor does it yield such principles by a direct adaptation of the methods used in [14, 20, 10, 17]. We believe however that the methods introduced here will play an important role when one tries to generalize the result to arbitrary, regular cardinals κ. We close the introduction with a brief description of the paper. Section 2 fixes the generalized descriptive set theoretic notation. Section 3 introduces E-complete and E-proper forcings and records the preservation facts used in the construction. Section 4 builds the Cohen reservoirs and the definable Suslin-tree apparatus. Section 5 defines the direct container coding. Section 6 introduces clean allowability and the separation iteration. Section 7 proves the separation theorem and verifies the final cardinal arithmetic. The paper ends with a short list of open questions.

2 Generalized descriptive set theory preliminaries

We work on 2^{ω_1} with the bounded topology. If $s \in 2^{<\omega_1}$, then

$$N_s = \{x \in 2^{\omega_1} : s \subseteq x\}$$

is basic open. A tree on $(2^{<\omega_1})^n$ is a downward closed set of tuples of equal length. If T is a tree on $(2^{<\omega_1})^{n+1}$, let

$$p[T] = \{\vec{x} \in (2^{\omega_1})^n : \exists y \in 2^{\omega_1} \ ((\vec{x}, y) \in [T])\}.$$

In the final model we will have CH, hence

$$2^{<\omega_1} = \omega_1.$$

Thus arbitrary trees on $2^{<\omega_1}$ have size at most ω_1 and can be coded by elements of 2^{ω_1}. Consequently the standard tree-projection presentation of Σ^1_1 agrees with the syntactic boldface presentation by fixed lightface closed matrices and parameters from 2^{ω_1}. For bookkeeping purposes we use the syntactic presentation. Fix a recursive enumeration of the lightface closed matrices. If $m < \omega$ is a code and $p \in 2^{\omega_1}$, write

$$M_m(x, p)$$

for the corresponding $\Sigma^1_1(p)$ assertion, i.e. for a formula of the form

$$(\exists y \in 2^{\omega_1})\, \theta_m(x, y, p),$$

where θ_m is closed. The boldface classes are obtained by allowing the parameter $p \in 2^{\omega_1}$. Since CH holds in the final model, this convention is equivalent to the usual boldface tree-parametric one. For the general correspondence between tree projections and $\Sigma_1(H(\kappa^+))$ definitions, including its forcing-absoluteness features, see [30]. For the Borel/Borel* side of generalized Baire space and infinitary-language variants, see [7, 25].

Definition 2.1. *The Σ^1_1-separation property on 2^{ω_1} says that whenever $A_0, A_1 \subseteq 2^{\omega_1}$ are disjoint boldface Σ^1_1 sets, there is a boldface Δ^1_1 set $D \subseteq 2^{\omega_1}$ such that*

$$A_0 \subseteq D \quad \textit{and} \quad D \cap A_1 = \emptyset.$$

3 E-complete forcings

The no-new-real part of the construction is organized around a fixed stationary co-stationary set $E \subseteq \omega_1$. This is the set of good heights used in all fusion arguments. This stationary-set relativization of properness and club-shooting preservation is in the tradition of the standard club-shooting and proper-forcing machinery [4, 1, 34], and of the earlier stationary-coding arguments [11, 24].

M-generic decreasing sequences. Let $M \prec H(\Theta)$ be countable and let $\mathbb{Q} \in M$. A decreasing sequence

$$q_0 \geq q_1 \geq q_2 \geq \cdots$$

of conditions in $M \cap \mathbb{Q}$ is called *M-generic* if the filter on $M \cap \mathbb{Q}$ generated by the sequence is generic over M. Equivalently, for every dense open set $D \subseteq \mathbb{Q}$ with $D \in M$, there is $n < \omega$ such that $q_n \in D$. Equivalently, for every maximal antichain $A \in M$ of $\mathbb{Q}$, there are $n < \omega$ and $a \in A \cap M$ such that

$$q_n \leq a.$$

Definition 3.1 (E-complete forcing)**.** *Let $E \subseteq \omega_1$ be stationary. A forcing $\mathbb{Q}$ is* E-complete *if for every sufficiently large regular Θ, every countable elementary submodel*

$$M \prec H(\Theta)$$

with $\mathbb{Q} \in M$ and $M \cap \omega_1 \in E$, and every decreasing sequence

$$q_0 \geq q_1 \geq q_2 \geq \cdots$$

of conditions in $M \cap \mathbb{Q}$ which is M-generic, there is a condition $q_\omega \in \mathbb{Q}$ below all q_n.

Definition 3.2 (E-proper forcing). *Let $E \subseteq \omega_1$ be stationary. A forcing $\mathbb{Q}$ is* E-proper *if for every sufficiently large regular Θ, every countable elementary submodel $M \prec H(\Theta)$ with $\mathbb{Q} \in M$ and $M \cap \omega_1 \in E$, and every $q \in M \cap \mathbb{Q}$, there is a condition $q^* \leq q$ which is $(M, \mathbb{Q})$-generic.*

Lemma 3.3. *If $\mathbb{Q}$ is E-complete, then $\mathbb{Q}$ is E-proper and adds no new reals. In particular, $\mathbb{Q}$ preserves ω_1 and preserves every stationary subset of E.*

Proof. Let $M \prec H(\Theta)$ be countable with $\mathbb{Q} \in M$ and $\delta = M \cap \omega_1 \in E$, and let $q \in M \cap \mathbb{Q}$. Enumerate in M the maximal antichains of $\mathbb{Q}$ which belong to M as $\langle A_n : n < \omega \rangle$. Working inside M, build a decreasing sequence

$$q = q_0 \geq q_1 \geq q_2 \geq \cdots$$

as follows. Since A_n is maximal and $q_n \in M$, elementarity gives $a_n \in A_n \cap M$ which is compatible with q_n. Again by elementarity, choose $q_{n+1} \in M \cap \mathbb{Q}$ such that

$$q_{n+1} \leq q_n, a_n.$$

Then $q_{n+1} \leq a_n$, and hence the sequence meets the dense open set generated by A_n. Thus the sequence is M-generic. By E-completeness there is a lower bound $q^* \leq q_n$ for all $n < \omega$. Then q^* is $(M, \mathbb{Q})$-generic. Thus $\mathbb{Q}$ is E-proper. The preservation of ω_1 and of stationary subsets of E is the usual properness argument, restricted to the stationary collection of countable models whose height lies in E. To see that no reals are added, let $\dot{r}$ be a $\mathbb{Q}$-name for an element of 2^ω and let $q \in \mathbb{Q}$. Choose $M \prec H(\Theta)$ countable with $q, \mathbb{Q}, \dot{r} \in M$ and $M \cap \omega_1 \in E$. Inside M, build an M-generic decreasing sequence

$$q = q_0 \geq q_1 \geq q_2 \geq \cdots$$

so that q_n decides $\dot{r}(n)$. Let q_ω be a lower bound. Then q_ω decides every bit of $\dot{r}$, hence forces $\dot{r}$ to be a ground model real. □

Lemma 3.4. *Let $A \subseteq \omega_1$ contain E, and let* CU(A) *be the forcing of closed bounded subsets of A, ordered by end-extension. Then* CU(A) *is E-complete.*

Proof. Let $M \prec H(\Theta)$ be countable with $\delta = M \cap \omega_1 \in E$, and let

$$c_0 \geq c_1 \geq c_2 \geq \cdots$$

be an M-generic decreasing sequence in $\mathrm{CU}(A) \cap M$. The union $\bigcup_n c_n$ is a closed bounded subset of δ cofinal in δ. Since $\delta \in E \subseteq A$, the set

$$c_\omega = \bigcup_n c_n \cup \{\delta\}$$

is a closed bounded subset of A and is a lower bound for the sequence. □

Lemma 3.5. *Let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$$

be a countable-support iteration. Suppose that for every $\alpha < \gamma$,

$$\mathbb{P}_\alpha \Vdash \text{“}\dot{\mathbb{Q}}_\alpha \text{ is } E\text{-complete.”}$$

Then $\mathbb{P}_\gamma$ is E-complete. In particular $\mathbb{P}_\gamma$ is E-proper and adds no reals.

Proof. This is the standard fusion proof for E-complete forcings. We recall the argument to fix the form used later. Let $M \prec H(\Theta)$ be countable with $M \cap \omega_1 \in E$, and let

$$p_0 \geq p_1 \geq p_2 \geq \cdots$$

be an M-generic decreasing sequence in $M \cap \mathbb{P}_\gamma$. Let $\delta_M = M \cap \gamma$. The union of the supports of the p_n's is countable and contained in $M \cap \gamma$. We define a lower bound p_ω on this support by induction on the support order. At coordinate α, the initial segment $p_\omega \upharpoonright \alpha$ is already a lower bound for the corresponding initial segments, and in the $\mathbb{P}_\alpha$-extension the sequence of α-coordinates is an $M[G_\alpha]$-generic decreasing sequence in the interpreted iterand. Since the iterand is forced to be E-complete and $M[G_\alpha] \cap \omega_1 = M \cap \omega_1 \in E$, there is a lower bound for the coordinate sequence. Placing these lower bounds at all coordinates in the countable support gives a condition $p_\omega \in \mathbb{P}_\gamma$ below all p_n. □

Theorem 3.6 (Abraham–Shelah theorem, E-proper form)**.** *Assume CH. Let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$$

be a countable-support iteration with $\gamma \leq \omega_2$ such that

$$\mathbb{P}_\alpha \Vdash \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is } E\textit{-proper and } |\dot{\mathbb{Q}}_\alpha| \leq \aleph_1\text{”}$$

for every $\alpha < \gamma$. Then $\mathbb{P}_\gamma$ is E-proper and satisfies the ω_2-chain condition. Moreover, every bounded intermediate extension $V^{\mathbb{P}_\alpha}$, $\alpha < \omega_2$, satisfies CH.

Proof. This is the standard E-proper version of Shelah's preservation theorem for countable-support iterations of size-$\aleph_1$ forcings under CH. The proof is the same as in the proper case presented by Abraham [1, Theorem 2.10 and the subsequent CH preservation theorem], with the elementary submodels restricted to the stationary class

$$\{M \prec H(\Theta) : M \cap \omega_1 \in E\}.$$

We use the theorem in this form. □

Theorem 3.7 (Miyamoto preservation, E-proper form)**.** *Let $E \subseteq \omega_1$ be stationary, let T be a Suslin tree, and let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$$

be a countable-support iteration. Suppose that for every $\alpha < \gamma$,

$$\mathbb{P}_\alpha \Vdash \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is } E\textit{-proper and preserves } T \textit{ as a Suslin tree.”}$$

Then $\mathbb{P}_\gamma$ preserves T as a Suslin tree. More generally, the same conclusion holds uniformly for a fixed family of Suslin trees, provided each iterand preserves every tree in the family.

Proof. This is the E-proper analogue of Miyamoto's countable-support preservation theorem for Suslin trees. The proper case is due to Miyamoto [34]. The E-proper formulation used here is the standard relativization to countable elementary submodels M with $M \cap \omega_1 \in E$; see the earlier preservation argument [11]. Equivalently, one uses the Miyamoto criterion with “proper” replaced by “E-proper” and checks $(M, \mathbb{P} \times T)$-genericity only for such models M. □

4 The preliminary model

We now build the model over which the separation iteration is performed. The preliminary forcing has three jobs: reserve the good stationary set E, add the ω_1-Cohen reservoirs whose initial-segment traces will be used for bookkeeping, and add a uniformly $\Sigma_1(H(\omega_2), \omega_1)$-definable independent sequence of Jech-generic Suslin trees. The background coding tools are almost-disjoint and stationary coding, together with Jech-style Suslin-tree generics [27, 26, 9, 11, 20]. The stationary coding is used only to make the Suslin-tree apparatus definable, and it is arranged to respect E. The Cohen reservoirs remain generic auxiliary traces; their individual membership relation is not part of the final projective decoding.

4.1 The reserved stationary set

Work in L. Fix a stationary co-stationary set

$$E \subseteq \omega_1$$

from L, and put

$$E^* = \omega_1 \setminus E.$$

We shall use stationary coding only on subsets of E^*, so that all club-shooting targets contain E. We also fix, in L, a family

$$\langle E_\zeta : \zeta < \omega_2 \rangle$$

of stationary subsets of E^* which are pairwise almost disjoint modulo the nonstationary ideal. In fact we choose the family so that distinct members have only countable intersection.

Here is the standard construction. Since $V = L$, Jensen's diamond principle holds on every stationary subset of ω_1. Fix a $\diamondsuit(E^*)$- sequence

$$\langle D_\alpha : \alpha \in E^* \rangle.$$

Thus, for every $X \subseteq \omega_1$, the set

$$A_X = \{\alpha \in E^* : D_\alpha = X \cap \alpha\}$$

is stationary. Using the canonical well-order of L, fix a sequence

$$\langle X_\zeta : \zeta < \omega_2 \rangle$$

of pairwise distinct subsets of ω_1, and define

$$E_\zeta = A_{X_\zeta}.$$

Then every E_ζ is stationary and contained in E^*. Moreover, if $\zeta \neq \xi$, then $E_\zeta \cap E_\xi$ is countable. Indeed, let

$$\beta = \min(X_\zeta \triangle X_\xi).$$

For every $\alpha \in E^*$ with $\alpha > \beta$, we have

$$X_\zeta \cap \alpha \neq X_\xi \cap \alpha,$$

and hence α cannot belong to both E_ζ and E_ξ. Therefore

$$E_\zeta \cap E_\xi \subseteq \beta + 1,$$

which is countable.

Thus

$$\langle E_\zeta : \zeta < \omega_2 \rangle$$

is not a partition of $\omega_1 \setminus E$, but rather a uniformly definable almost disjoint stationary family inside $\omega_1 \setminus E$. This is the family used by the stationary coding. Consequently every club-shooting target contains E.

4.2 Reservoir sets and Suslin trees

Let

$$\mathrm{Add}_{\omega_1}(\omega_2)$$

denote the standard forcing, computed in L, whose conditions are countable partial functions from $\omega_2 \times \omega_1$ to 2, ordered by reverse inclusion. Let

$$\vec{C} = \langle C_\nu : \nu < \omega_2 \rangle$$

be the sequence of ω_1-Cohen subsets of ω_1 added by this forcing. It is countably closed, adds no reals, and under CH has size ω_2. Fix in L a bijection

$$\vartheta : (2^{<\omega_1})^L \longrightarrow \omega_1$$

with branch-cofinal image, for example with $\mathrm{lh}(s) \leq \vartheta(s)$ for all $s \in (2^{<\omega_1})^L$. For $y \in 2^{\omega_1}$ define its initial-segment trace by

$$\mathrm{Tr}(y) = \{\vartheta(y \restriction \alpha) : 0 < \alpha < \omega_1\}.$$

For each reservoir coordinate set

$$T_\nu = \{\vartheta(C_\nu \restriction \alpha) : 0 < \alpha < \omega_1\}.$$

The raw Cohen sets C_ν provide homogeneity; the actual coding traces are the almost disjoint sets T_ν. Let $\mathbb{J}$ be Jech's forcing for adding an ω_1-Suslin tree [26]. We use the standard countable-support product

$$\mathbb{J}_{\omega_2} = \prod_{\xi<\omega_2}^{\mathrm{cs}} \mathbb{J}_\xi.$$

Let

$$\vec{S} = \langle S^i_\xi : i < 2, \xi < \omega_2 \rangle$$

be the resulting sequence of pairs of generic Suslin trees, obtained by a fixed constructible reindexing of the ω_2 many coordinates.

Lemma 4.1. *Let $\dot{S}$ be the $\mathbb{J}$-name for the generic Suslin tree. For every $0 < n < \omega$, the two-step forcing*

$$\mathbb{J} * \dot{S}^n$$

has a σ-closed dense subset. Consequently,

$$\mathbb{J} \Vdash \text{“}\dot{S}^n \text{ is } \omega\text{-distributive.”}$$

Proof. Let D_n consist of all pairs

$$(t, \langle s_0, \ldots, s_{n-1} \rangle)$$

where t is a countable normal tree of successor height and $s_0, \ldots, s_{n-1}$ are nodes on the top level of t. This is dense in $\mathbb{J} * \dot{S}^n$: first decide the finitely many branch nodes mentioned by the second coordinate and then end-extend the tree so that these nodes have extensions on the new top level. If

$$(t_m, \langle s^m_0, \ldots, s^m_{n-1} \rangle) \qquad (m < \omega)$$

is decreasing in D_n, take the union of the t_m's and add a new top level carrying the limit nodes

$$s_j = \bigcup_m s^m_j \qquad (j < n).$$

The resulting condition is a lower bound. Thus D_n is σ-closed. □

Lemma 4.2. *Assume CH. Let I be a set of cardinality at most ω_1, and let $m : I \to \omega \setminus \{0\}$ be a finite multiplicity function. Let*

$$\mathbb{J}_I = \prod_{\iota \in I}^{\text{cs}} \mathbb{J}_\iota$$

and let $\dot{S}_\iota$ be the ι-th generic Suslin tree. In the $\mathbb{J}_I$-extension put

$$\dot{\mathbb{B}}(I, m) = \prod_{\iota \in I}^{\text{cs}} \dot{S}_\iota^{m(\iota)}.$$

Then

$$\mathbb{J}_I * \dot{\mathbb{B}}(I, m)$$

has a σ-closed dense subset. Consequently

$$\mathbb{J}_I \Vdash \text{“}\dot{\mathbb{B}}(I, m) \text{ is } \omega\text{-distributive.”}$$

Under CH, $\dot{\mathbb{B}}(I, m)$ has size $\aleph_1$.

Proof. A dense condition consists of a countable support $a \subseteq I$, a Jech tree approximation $p(\iota)$ for each $\iota \in a$, and for each $\iota \in a$ an $m(\iota)$-tuple of nodes on the top level of $p(\iota)$. Density is obtained coordinate by coordinate as in Lemma 4.1. For a countable descending sequence, take the union of the supports and, at each coordinate, take the union of the tree approximations and add the corresponding limit top-level nodes. The support remains countable, so this gives a lower bound.

The final size estimate follows from CH: there are only $\aleph_1$ many countable supports in I, and at each coordinate only $\aleph_1$ many possible finite tuples of tree nodes. □

4.3 Stationary coding respecting E

Only the Suslin-tree apparatus must be uniformly definable over $H(\omega_2)$ from ω_1. The Cohen reservoir sequence $\vec{C}$ is kept as a generic auxiliary sequence and is not stationary-coded. We code the membership relation of $\vec{S}$ by shooting clubs through complements of the stationary sets E_ζ. Since each E_ζ is disjoint from E, every target set contains E, and therefore each club-shooting step is E-complete by Lemma 3.4. We now define the preliminary forcing formally. Let

$$\mathbb{P}_{\text{res}} = \text{Add}^L_{\omega_1}(\omega_2)$$

be the reservoir forcing defined above, and let

$$\mathbb{P}_{\text{J}} = \mathbb{J}_{\omega_2} = \prod_{\xi < \omega_2}^{\text{cs}} \mathbb{J}_\xi$$

be the countable-support product of Jech forcings, computed in L. Put

$$\mathbb{P}_{\text{base}} = \mathbb{P}_{\text{res}} \times \mathbb{P}_{\text{J}}.$$

The symbols $\vec{C}$ and $\vec{S}$ are the canonical $\mathbb{P}_{\text{base}}$-names for the corresponding reservoir sequence and Suslin-tree sequence. Since CH holds in L, fix an L-definable bijection

$$\theta : 2 \times \omega_2 \times (2^{<\omega_1})^L \longrightarrow \omega_2.$$

In a $\mathbb{P}_{\text{base}}$-extension, define

$$A_{\text{prep}} = \{\theta(i, \xi, s) : i < 2,\ \xi < \omega_2,\ s \in S^i_\xi\}.$$

Let $\dot{A}_{\text{prep}}$ be the canonical $\mathbb{P}_{\text{base}}$-name for this set. Over the $\mathbb{P}_{\text{base}}$-extension, let

$$\mathbb{P}_{\text{stat}} = \mathbb{P}_{\text{stat}}(\dot{A}_{\text{prep}})$$

be the countable-support iteration

$$\langle \mathbb{P}^\zeta_{\text{stat}}, \dot{\mathbb{R}}^\zeta_{\text{stat}} : \zeta < \omega_2 \rangle$$

which, at stage ζ, uses

$$\dot{\mathbb{R}}^\zeta_{\text{stat}} = \begin{cases} \mathrm{CU}(\omega_1 \setminus E_\zeta), & \text{if } \zeta \in A_{\text{prep}}, \\ \mathbf{1}, & \text{if } \zeta \notin A_{\text{prep}}. \end{cases}$$

Finally set

$$\mathbb{P}_{\text{prep}} = \mathbb{P}_{\text{base}} * \dot{\mathbb{P}}_{\text{stat}}.$$

Thus, in the preliminary extension,

$$\zeta \in A_{\text{prep}} \quad \Longleftrightarrow \quad E_\zeta \text{ is nonstationary.}$$

Indeed, the forward direction is forced by shooting a club through $\omega_1 \setminus E_\zeta$. Conversely, if $\zeta \notin A_{\text{prep}}$, then all other club-shooting coordinates have targets which contain all but countably many points of E_ζ, by the almost-disjointness of the family $\langle E_\xi : \xi < \omega_2 \rangle$. The standard countable-support almost-disjoint stationary-coding argument therefore preserves the stationarity of E_ζ. Let

$$G_{\text{prep}} = G_{\text{base}} * G_{\text{stat}}$$

be $\mathbb{P}_{\text{prep}}$-generic over L, with G_{base} generic for $\mathbb{P}_{\text{base}}$ and G_{stat} generic for the stationary-coding tail. Put

$$W = L[G_{\text{prep}}].$$

Lemma 4.3 (Stationary coding preserves the tree apparatus)**.** *The stationary coding stage used in the definition of W preserves the Suslin apparatus added by the preliminary Jech product. More precisely, in W:*

- (i) *the sequence $\vec{S}$ remains independent;*
- (ii) *each unused tree coordinate remains Suslin;*
- (iii) *the finite off-the-generic-branches property remains true;*
- (iv) *the E-good-level property remains true: if $\delta \in E$ and $\langle s_n : n < \omega \rangle$ is an increasing sequence of nodes in one of the trees with heights cofinal in δ, then $\bigcup_n s_n$ is a node on level δ; and*
- (v) *the reserved set E remains stationary.*

Proof. Each stationary-coding component is a club-shooting forcing $\mathrm{CU}(A)$ with $E \subseteq A$. By Lemma 3.4, such a forcing is E-complete. Hence the stationary-coding iteration is E-complete by Lemma 3.5; in particular it is E-proper, preserves the stationarity of E, and adds no reals. It remains to record the preservation of the tree apparatus. Work in the model after the reservoir forcing and the product of Jech forcings, but before the stationary coding. Fix finitely many tree coordinates and finitely many generic branches through them. The dense presentation from Lemmas 4.1 and 4.2 gives the usual fusion lower bound by adding the model height $M \cap \omega_1$ as a new top level of the relevant tree approximations. Since all stationary-coding targets contain E, the same lower bound simultaneously extends the club-shooting coordinates whenever $M \cap \omega_1 \in E$. Thus the stationary-coding iteration is compatible with the Jech-tree fusion argument. It cannot add a cofinal branch through an unused Jech-generic tree, and it cannot destroy the finite off-the-generic-branches property. The same fusion explicitly adds the limit node at every good model height in E, so the E-good-level property is preserved as well. The independence of $\vec{S}$ is preserved for the same reason, applied to finite products of distinct tree coordinates. □

Lemma 4.4. *In W the following hold.*

(i) $2^\omega = \omega_1$ *and* $2^{\omega_1} = \omega_2$.

(ii) *The sequence* $\vec{S} = \langle S^i_\xi : i < 2, \xi < \omega_2 \rangle$ *is uniformly* $\Sigma_1(H(\omega_2), \omega_1)$*-definable. The reservoir sequence* $\vec{C} = \langle C_\nu : \nu < \omega_2 \rangle$ *is the* $\mathrm{Add}^L_{\omega_1}(\omega_2)$*-generic sequence added in the preliminary forcing, and the associated traces are* T_ν*. Neither* $\vec{C}$ *nor the individual traces* T_ν *are part of the stationary code.*

(iii) *The sequence* $\vec{S}$ *is independent: finite products of distinct members are Suslin.*

(iv) *Each* S^i_ξ *has the finite off-the-generic-branches property.*

(v) *The sequence* $\vec{S}$ *has the* E*-good-level property: whenever* $\delta \in E$ *and* $\langle s_n : n < \omega \rangle$ *is an increasing sequence of nodes in some* S^i_ξ *with heights cofinal in* δ*, the union* $\bigcup_n s_n$ *is a node of* S^i_ξ *at level* δ.

(vi) *The reserved set* E *remains stationary.*

Proof. The reservoir forcing and the product of Jech forcings add no reals. The stationary coding uses only club-shooting forcings through sets containing E, so it is E-complete and adds no reals. Thus CH is preserved. The reservoir forcing adds ω_2 many distinct subsets of ω_1, and the preliminary forcing has size ω_2 and the usual ω_2-chain-condition/size control under GCH in L, so $2^{\omega_1} = \omega_2$. The definability of $\vec{S}$ follows from the stationary coding: for the fixed coding map θ, membership $s \in S^i_\xi$ is equivalent to the assertion that $E_{\theta(i,\xi,s)}$ is nonstationary, i.e. to the $\Sigma_1(H(\omega_2), \omega_1)$ statement that there is a club disjoint from $E_{\theta(i,\xi,s)}$. No corresponding definition of the Cohen reservoirs is needed. The preservation of the Suslin apparatus, including the finite off-branch and E-good-level properties, is Lemma 4.3. The independence and finite off-branch properties used there come from the product Jech lemma and the Fuchs–Hamkins off-branch theorem [9, 20]. □

Remark 4.5. The important point is that E is not used for coding. All stationary sets whose stationarity is killed or preserved are subsets of $\omega_1 \setminus E$. Thus whenever a countable model has height in E, the usual fusion lower bound can add that height as a new top point in the relevant club-shooting conditions and as a new top level in the relevant Jech-tree approximations.

5 Direct container coding

We now define the direct coding blocks. A container is an ω_1-sized subset of ω_2, partitioned into ω_1 many ω_1-blocks. A fresh ω_1-Cohen reservoir set determines an almost-disjoint derived trace, that trace chooses which blocks of the container are used, and in each selected block the forcing writes the full characteristic function of the relevant ω_1-code. Direct coding by almost-disjoint traces and Suslin-tree branch patterns is the generalized analogue of the projective coding machinery used in [27, 5, 12, 14, 10, 13].

5.1 Containers and derived reservoir traces

Fix in L a partition

$$\langle I_\eta : \eta < \omega_2 \rangle$$

of ω_2 into sets of size ω_1. Each I_η is called a container. Fix uniform L-definable bijections

$$e_\eta : \omega_1 \times \omega_1 \longrightarrow I_\eta.$$

For $\gamma < \omega_1$ put

$$I_{\eta,\gamma} = \{e_\eta(\gamma, \alpha) : \alpha < \omega_1\}.$$

Thus

$$I_\eta = \bigcup_{\gamma<\omega_1} I_{\eta,\gamma}$$

is partitioned into ω_1 many pieces, each of size ω_1. For $i < 2$, $\eta < \omega_2$, $\gamma < \omega_1$ and $\alpha < \omega_1$, set

$$S^i_{\eta,\gamma,\alpha} = S^i_{e_\eta(\gamma,\alpha)}.$$

The pair

$$(S^0_{\eta,\gamma,\alpha}, S^1_{\eta,\gamma,\alpha})$$

is the tree pair at bit coordinate α inside the γ-th block of the container I_η. The reservoir sequence

$$\vec{C} = \langle C_\nu : \nu < \omega_2 \rangle$$

consists of the ω_1-Cohen subsets of ω_1 added by the preliminary forcing. The construction does not use C_ν itself as the set of coding blocks. Instead it uses the derived initial-segment trace

$$T_\nu = \{\vartheta(C_\nu \restriction \xi) : 0 < \xi < \omega_1\} \subseteq \omega_1,$$

where the bijection $\vartheta : (2^{<\omega_1})^L \to \omega_1$ was fixed in the preliminary construction. A use of the container I_η chooses one reservoir index $\nu < \omega_2$. The derived trace T_ν determines the blocks of I_η in which the code is written: for every $\gamma \in T_\nu$, the whole code is written into the block $I_{\eta,\gamma}$. We shall use the following elementary consequence of replacing a Cohen reservoir by the set of its coded initial segments.

Lemma 5.1. *In the preliminary model the sequence*

$$\langle T_\nu : \nu < \omega_2 \rangle$$

consists of unbounded subsets of ω_1 *and is pairwise almost disjoint in the strong sense that*

$$T_\nu \cap T_\mu$$

is bounded in ω_1 *whenever* $\nu \neq \mu$. *Consequently, if* $F \subseteq \omega_2 \setminus \{\nu\}$ *is finite, then*

$$T_\nu \setminus \bigcup_{\mu\in F} T_\mu$$

is unbounded in ω_1.

Proof. Each T_ν is unbounded because $\mathrm{lh}(s) \leq \vartheta(s)$ for all $s \in (2^{<\omega_1})^L$ and the initial segments $C_\nu \restriction \xi$ are pairwise distinct. If $\nu \neq \mu$, then the two Cohen sets C_ν and C_μ are distinct. Let $\delta < \omega_1$ be the first coordinate at which they differ. Then the only common initial segments of C_ν and C_μ have length at most δ. Since ϑ is injective,

$$T_\nu \cap T_\mu \subseteq \{\vartheta(C_\nu \restriction \xi) : 0 < \xi \leq \delta\},$$

which is countable, hence bounded in ω_1. The final assertion follows because a finite union of bounded subsets of ω_1 is bounded. □

5.2 The tuple code

For each separation tuple

$$t = (\rho, x, m, k, p, i)$$

with $\rho < \omega_2$, $x, p \in 2^{\omega_1}$, $m, k < \omega$ and $i < 2$, fix a canonical code

$$X_t \subseteq \omega_1.$$

The coding map is injective and uniformly continuous in the 2^{ω_1} coordinates. For example one may interleave the characteristic functions of x and p with fixed codes for ρ, m, k, i using a constructible bijection $\omega_1 \cong \omega_1 \times \omega$. Only injectivity is needed in the no-unwanted-code argument, because the derived reservoir traces are pairwise almost disjoint.

Definition 5.2 (Direct coding block)**.** *Let $t = (\rho, x, m, k, p, i)$ be a separation tuple, let $\eta < \omega_2$ be a container, and let $\nu < \omega_2$ be a reservoir index. The direct coding forcing*

$$\mathbb{B}_{\eta,\nu,t}$$

adds, for every $\gamma \in T_\nu$ and every $\alpha < \omega_1$, a cofinal branch through

$$S^{X_t(\alpha)}_{\eta,\gamma,\alpha}.$$

A condition is a countable partial function whose domain is a countable subset of

$$\{(\gamma, \alpha) : \gamma \in T_\nu,\ \alpha < \omega_1\}$$

and which assigns to (γ, α) a node of $S^{X_t(\alpha)}_{\eta,\gamma,\alpha}$, ordered by coordinatewise extension.

More generally, if a finite list of tuples

$$t_0, \ldots, t_{r-1}$$

is coded into the same container I_η, we use finitely many derived reservoir traces

$$T_{\nu_0}, \ldots, T_{\nu_{r-1}}$$

and force with the countable-support product of the blocks

$$\mathbb{B}_{\eta,\nu_\ell,t_\ell} \qquad (\ell < r).$$

We require the reservoir indices ν_ℓ to be distinct for the distinct uses of the same container. The derived traces may have bounded overlap, but they are pairwise almost disjoint by Lemma 5.1. Decoding will look for a cofinal subtrace on which no other use of the same container is active.

Lemma 5.3. *Assume CH in the ambient model and suppose the Suslin-tree apparatus satisfies the E-good-level property from Lemma 4.4. Every direct coding block $\mathbb{B}_{\eta,\nu,t}$ is E-complete and has size $\aleph_1$. The same holds for any finite product of direct coding blocks using one container and finitely many derived reservoir traces. Hence these blocks are E-proper and add no reals.*

Proof. The size is $\aleph_1$ under CH: a condition has countable support in a set of size $\aleph_1$, and at each coordinate the relevant tree has size $\aleph_1$. We prove E-completeness. Let $M \prec H(\Theta)$ be countable with $\delta = M \cap \omega_1 \in E$, and let

$$q_0 \geq q_1 \geq q_2 \geq \cdots$$

be an M-generic descending sequence in the direct coding block. The union of the supports is countable. Fix a coordinate (γ, α) in this union. The sequence of nodes assigned at this coordinate is increasing in the corresponding Jech-generic Suslin tree. If the heights are cofinal in δ, then by the good-level property at heights in E, the union of this increasing sequence is a node on level δ of that tree. If the heights are bounded below δ, first take the eventual union and then extend it to level δ using normality of the tree. Put this limit node into the coordinate. Doing this simultaneously for all coordinates in the countable union of supports gives a condition below every q_n. The argument for finitely many traces is identical, since the union of finitely many countable supports is countable. □

Remark 5.4. Lemma 5.3 is the quotient form of the dense σ-closed presentation in Lemma 4.2. In the combined Jech-plus-branch forcing, the lower bound is obtained by end-extending the tree approximations and adding limit top-level nodes. After the preliminary Jech and stationary-coding forcing has been performed, the same construction is reflected in the E-good-level property of the fixed tree sequence.

5.3 Preservation of unused tree coordinates

The proof of the no-unwanted-code lemma uses a preservation fact about tree coordinates which are not deliberately selected by a direct coding block. We isolate this fact here so that the later argument is only a decoding argument.

Definition 5.5. *Let $\mathbb{B}_{\eta,\nu,t}$ be a direct coding block. Its tree-coordinate support is*

$$\mathrm{supp}_{\mathrm{tr}}(\mathbb{B}_{\eta,\nu,t}) = \{(X_t(\alpha), e_\eta(\gamma, \alpha)) : \gamma \in T_\nu, \alpha < \omega_1\}.$$

Equivalently, $\mathbb{B}_{\eta,\nu,t}$ uses exactly the trees

$$S^{X_t(\alpha)}_{\eta,\gamma,\alpha} \qquad (\gamma \in T_\nu,\ \alpha < \omega_1).$$

A countable-support partial run uses a tree coordinate (i, τ) if some iterand in the run has (i, τ) in its tree-coordinate support.

Lemma 5.6 (Jech sealing and off-branch preservation)**.** *Let K be a set of tree coordinates of size at most ω_1, and let $\tau = (i, \nu)$ be a tree coordinate not in K. Force first with the product of Jech forcings adding the trees indexed by $K \cup \{\tau\}$ and then add, with countable support, the branches through the coordinates in K specified by a finite multiplicity function on each coordinate. In the resulting extension, the tree S^i_ν remains Suslin.*

More generally, if finitely many branches through S^i_ν are deliberately added first, then every subtree of S^i_ν off the union of those finitely many branches remains Suslin after the remaining branch forcings on K.

Proof. Write

$$m : K \longrightarrow \omega$$

for the finite multiplicity function, and let

$$\mathbb{R}(K, m, \tau) = \Big(\prod_{\sigma \in K \cup \{\tau\}}^{\text{cs}} \mathbb{J}_\sigma \Big) * \dot{\mathbb{B}}(K, m),$$

where

$$\dot{\mathbb{B}}(K, m) = \prod_{\sigma \in K}^{\text{cs}} \prod_{j < m(\sigma)} \dot{S}_\sigma.$$

Thus no branch coordinate over τ occurs in $\dot{\mathbb{B}}(K, m)$. It is enough to prove

$$\mathbb{R}(K, m, \tau) \Vdash \text{“}S^i_\nu \text{ is Suslin.”}$$

Fix $p \in \mathbb{R}(K, m, \tau)$ and a name $\dot{A}$ such that

$$p \Vdash \text{“}\dot{A} \subseteq S^i_\nu \text{ is a maximal antichain.”}$$

Choose a countable

$$M \prec H(\Theta) \quad \text{with} \quad p, \dot{A}, K, m, \tau, \mathbb{R}(K, m, \tau) \in M,$$

where Θ is large, and put $\delta = M \cap \omega_1$. Work in the dense σ-closed presentation from Lemma 4.2. In this presentation the coordinates in K carry their finitely many branch markers, whereas the coordinate τ carries only the Jech-tree approximation. Construct an M-generic descending sequence

$$p = p_0 \geq p_1 \geq p_2 \geq \cdots$$

which meets all dense subsets of the presentation belonging to M and, for the τ-coordinate, meets the usual sealing requirements for $\dot{A}$. The latter requirements are the dense sets ensuring that, whenever a possible future δ-level node u at the τ-coordinate is produced by the fusion, some condition in the sequence decides some ground-model node $a \in S^i_\nu \cap M$ below u to be a member of $\dot{A}$. Let p_ω be the fusion of this sequence. Concretely, for $\sigma \in K$ we take the unions of the tree approximations and add the limit nodes determined by the finitely many branch markers. At the unused coordinate τ we take the union of the Jech approximations and add the sealing level $L_\tau \subseteq (S^i_\nu)_\delta$. The construction gives

$$p_\omega \Vdash (\forall u \in L_\tau)(\exists a \in S^i_\nu \cap M)\,(a \in \dot{A} \ \wedge \ a <_{S^i_\nu} u). \tag{$*$}$$

Every node of S^i_ν of height at least δ extends a unique member of L_τ. Hence p_ω forces that no element of $\dot{A}$ has height $> \delta$: if $b \in \dot{A}$ and $\mathrm{ht}(b) > \delta$, then $b \restriction \delta \in L_\tau$, and by $(*)$ there is $a \in S^i_\nu \cap M$ such that $a \in \dot{A}$ and

$$a <_{S^i_\nu} b \restriction \delta <_{S^i_\nu} b,$$

contradicting that $\dot{A}$ is an antichain. Thus

$$p_\omega \Vdash \dot{A} \subseteq S^i_\nu \restriction (\delta + 1).$$

The set $S^i_\nu \restriction (\delta + 1)$ is countable. Since p and $\dot{A}$ were arbitrary,

$$\mathbb{R}(K, m, \tau) \Vdash \text{“}S^i_\nu \text{ has no uncountable antichain.”}$$

Thus S^i_ν remains Suslin. For the off-branch statement, suppose first that $n < \omega$ branches

$$\dot b_0, \dots, \dot b_{n-1}$$

through S^i_ν are deliberately added. By the Fuchs–Hamkins off-branch preservation theorem [9, 20],

$$\mathbb{J}_\tau * \dot S^n_\tau \Vdash \text{“every subtree of } S^i_\nu \setminus \bigcup_{j<n} \dot b_j \text{ is Suslin.”}$$

Now work in the extension by these n branches and repeat the argument above with a name $\dot U$ for an off-branch subtree and a name $\dot A$ for a maximal antichain of $\dot U$. The fusion is carried out exactly as before, except that the sealing level is chosen inside $\dot U$. It yields

$$p_\omega \Vdash \dot A \subseteq \dot U \restriction (\delta + 1),$$

so the later branch forcings on the coordinates in K preserve the Suslinity of $\dot U$. This proves the stated off-branch preservation. □

Lemma 5.7. *Let $\mathbb{B}_{\eta,\nu,t}$ be a direct coding block and let (i, τ) be a tree coordinate which is not in $\mathrm{supp}_{\mathrm{tr}}(\mathbb{B}_{\eta,\nu,t})$. Then $\mathbb{B}_{\eta,\nu,t}$ preserves the Suslinity of S^i_τ.*

The same holds for any finite product of direct coding blocks, provided (i, τ) is outside the union of their tree-coordinate supports. If finitely many branches through S^i_τ have already been intentionally added, then the finite product preserves the Suslinity of every subtree of S^i_τ off those branches.

Proof. Let

$$K_0 = \mathrm{supp}_{\mathrm{tr}}(\mathbb{B}_{\eta,\nu,t}).$$

Since $(i, \tau) \notin K_0$, the pre-quotient forcing for this block has the form

$$\Big(\prod^{\mathrm{cs}}_{\sigma \in K_0 \cup \{(i,\tau)\}} \mathbb{J}_\sigma \Big) * \dot{\mathbb{B}}(K_0, 1),$$

where

$$\dot{\mathbb{B}}(K_0, 1) = \prod^{\mathrm{cs}}_{\sigma \in K_0} \dot S_\sigma.$$

The branch part is precisely the direct block $\mathbb{B}_{\eta,\nu,t}$ after passing to the quotient over the preliminary Jech and stationary-coding generics. By Lemma 5.6, the pre-quotient forcing satisfies

$$\Vdash \text{“} S^i_\tau \text{ is Suslin.”}$$

The quotient forcing theorem therefore gives, in the preliminary model W,

$$\mathbb{B}_{\eta,\nu,t} \Vdash \text{“} S^i_\tau \text{ is Suslin.”}$$

Equivalently, if some $b \in \mathbb{B}_{\eta,\nu,t}$ forced a name $\dot A$ to be an uncountable antichain of S^i_τ, then the corresponding pre-quotient condition would contradict the displayed forcing assertion. Now let

$$\mathbb{B}_* = \prod^{\mathrm{cs}}_{\ell<r} \mathbb{B}_{\eta,\nu_\ell,t_\ell}$$

be a finite product of direct coding blocks and put

$$K_* = \bigcup_{\ell<r} \mathrm{supp}_{\mathrm{tr}}(\mathbb{B}_{\eta,\nu_\ell,t_\ell}).$$

For $\sigma \in K_*$ define the finite multiplicity

$$m_*(\sigma) = \left|\{\ell < r : \sigma \in \mathrm{supp}_{\mathrm{tr}}(\mathbb{B}_{\eta,\nu_\ell,t_\ell})\}\right|.$$

Then $\mathbb{B}_*$ is the quotient interpretation of $\dot{\mathbb{B}}(K_*, m_*)$. If $(i,\tau) \notin K_*$, another application of Lemma 5.6 yields

$$\mathbb{B}_* \Vdash \text{“}S^i_\tau \text{ is Suslin.”}$$

The off-branch version is obtained by applying the off-branch part of Lemma 5.6 to the same pair (K_*, m_*): after the finitely many intentional branches through S^i_τ are factored off, the quotient product on K_* preserves every subtree disjoint from their union. □

Lemma 5.8. *Let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$$

be a clean countable-support partial run of direct coding blocks, with $\gamma \le \omega_2$. Suppose that a tree coordinate (i,ν) is not used by any iterand of the run. Then

$$\mathbb{P}_\gamma \Vdash \text{“}S^i_\nu \textit{ is Suslin.”}$$

More generally, suppose that the run deliberately adds only finitely many branches through S^i_ν. Then, after those branches are added, the remaining iteration preserves the Suslinity of every subtree of S^i_ν off their union.

Proof. By Lemma 5.7, each atomic direct coding block which avoids (i,ν) preserves S^i_ν. These iterands are E-proper and have size $\aleph_1$ by Lemmas 5.3 and 3.3. Therefore Theorem 3.7 implies that the whole partial run preserves the Suslinity of S^i_ν. For the off-branch version, first factor off the finitely many intentional branches through S^i_ν. Lemma 5.7 says that every later atomic block preserves the relevant off-branch subtree, and Theorem 3.7 again lifts this preservation through the countable-support iteration. □

5.4 Separation codes

Definition 5.9 (Direct separation code)**.** *Fix a container $\eta < \omega_2$. We write*

$$\mathrm{SepCoded}^\eta_\rho(x, m, k, p, i)$$

if there are a strictly increasing function $d : \omega_1 \to \omega_1$ and an ω_1-length code for a transitive model N such that N verifies the following.

(i) *The sequence $\vec{S}$ and the fixed container/block indexing are decoded according to the fixed $\Sigma_1(H(\omega_2), \omega_1)$ definitions.*

(ii) *The tuple $t = (\rho, x, m, k, p, i)$ has canonical bit code $X_t \subseteq \omega_1$.*

(iii) *The function d is strictly increasing. Hence its range is cofinal in ω_1.*

(iv) *For every $\beta < \omega_1$ and every $\alpha < \omega_1$, N contains a cofinal branch through*

$$S^{X_t(\alpha)}_{\eta,d(\beta),\alpha}.$$

We write $\mathrm{SepCoded}_\rho(x, m, k, p, i)$ if $\mathrm{SepCoded}^\eta_\rho(x, m, k, p, i)$ holds for some $\eta < \omega_2$.

If $B \subseteq \omega_2$ is a set of containers to be ignored, we write

$$\mathrm{SepCoded}^B_\rho(x, m, k, p, i)$$

if $\mathrm{SepCoded}^\eta_\rho(x, m, k, p, i)$ holds for some container $\eta \notin B$. In the final separator, B will be the activation-stage error term B_ρ, namely the set of all containers already used before the pair activated. Thus the decoder explicitly ignores all pre-activation coding noise.

Lemma 5.10. *For each $\rho < \omega_2$ and each parameter $B \in H(\omega_2)$ with $B \subseteq \omega_2$, the relation*

$$\mathrm{SepCoded}_\rho^B(x, m, k, p, i)$$

is Σ^1_1, uniformly in m, k, p, i and in a code for B. In particular, $\mathrm{SepCoded}_\rho(x, m, k, p, i)$ *is* Σ^1_1.

Proof. The witness is a subset of ω_1 coding the container η, the increasing cofinal trace d, and a well-founded extensional structure whose transitive collapse is the model N. The additional requirement for $\mathrm{SepCoded}_\rho^B$ is the bounded check $\eta \notin B$, using B as a boldface parameter. Since $B \in H(\omega_2)$, this is an allowable parameter in the $\Sigma_1(H(\omega_2))$ presentation of boldface Σ^1_1. The verification inside N is bounded: the preliminary tree apparatus $\vec{S}$ and the fixed container/block indexing are $\Sigma_1(H(\omega_2), \omega_1)$-definable, and the branch assertions are checked by the presence of the corresponding ω_1-sequences of tree nodes in N. The function d is an arbitrary cofinal trace; the definition does not require d to be a subtrace of any particular derived reservoir trace. Well-foundedness and extensionality are coded in the usual closed way by an ω_1-length witness. Thus the whole relation is a projection of a closed condition on an ω_1-code. □

Lemma 5.11. *Suppose a clean finite use of a container I_η codes the tuples $t_0, \ldots, t_{r-1}$ using pairwise distinct reservoir indices $\nu_0, \ldots, \nu_{r-1}$, and hence the pairwise almost-disjoint traces $T_{\nu_0}, \ldots, T_{\nu_{r-1}}$. Let $B \subseteq \omega_2$ with $\eta \notin B$. Then, after forcing with these direct coding blocks, each t_ℓ satisfies its corresponding direct separation code relative to B.*

Proof. Fix $\ell < r$. By Lemma 5.1, the set

$$T_{\nu_\ell} \setminus \bigcup_{j<r,\, j\neq\ell} T_{\nu_j}$$

is unbounded in ω_1. Let $d : \omega_1 \to \omega_1$ be a strictly increasing enumeration of an unbounded subset of this set. Along the range of d, no other use of the container is active. The direct coding block for t_ℓ therefore adds the required branches through

$$S^{X_{t_\ell}(\alpha)}_{\eta, d(\beta), \alpha}$$

for every $\beta, \alpha < \omega_1$. Since $\eta \notin B$, taking N to be the transitive collapse of a sufficiently large initial segment containing the relevant apparatus, the branches just added, and the function d gives a witness to the direct separation code relative to B. □

5.5 No unwanted direct codes

Lemma 5.12 (No unwanted direct codes)**.** *Let $\mathbb{P}$ be a clean countable-support partial run of direct coding blocks, and let $G \subseteq \mathbb{P}$ be generic. Fix a container $\eta < \omega_2$. If $W[G]$ satisfies*

$$\mathrm{SepCoded}_\rho^\eta(x, m, k, p, i),$$

then η was intentionally used in the run to code the tuple (ρ, x, m, k, p, i). Consequently, if $B \subseteq \omega_2$ and $\mathrm{SepCoded}_\rho^B(x, m, k, p, i)$ holds, then it is witnessed by an intentional code in some container outside B.

Proof. Let

$$t = (\rho, x, m, k, p, i)$$

and let X_t be its canonical code. Let $d : \omega_1 \to \omega_1$ be the cofinal trace appearing in the alleged code. Suppose first that the container η was not intentionally used. Choose any $\beta < \omega_1$ and any $\alpha < \omega_1$. The alleged code requires a branch through

$$S^{X_t(\alpha)}_{\eta,d(\beta),\alpha}.$$

This tree coordinate is unused by the partial run. By Lemma 5.8, it remains Suslin after forcing with $\mathbb{P}$. Therefore no cofinal branch through it can appear in $W[G]$, contradiction. Now suppose that η was intentionally used finitely many times, coding

$$t_0, \ldots, t_{r-1}$$

with reservoir indices

$$\nu_0, \ldots, \nu_{r-1}$$

and derived traces $T_{\nu_0}, \ldots, T_{\nu_{r-1}}$. If some value $d(\beta)$ is outside

$$\bigcup_{\ell<r} T_{\nu_\ell},$$

then the same unused-coordinate argument gives a contradiction. Hence the range of d is covered by this finite union of derived traces. Since d is cofinal, there is some $\ell < r$ such that

$$\{\beta < \omega_1 : d(\beta) \in T_{\nu_\ell}\}$$

is unbounded. By the almost-disjointness of the derived traces, we may choose $\beta < \omega_1$ such that

$$d(\beta) \in T_{\nu_\ell} \setminus \bigcup_{j<r,\, j\neq\ell} T_{\nu_j}.$$

Thus, at the block $d(\beta)$, the only intentional use of the container is the code for t_ℓ. If $t_\ell \neq t$, choose $\alpha < \omega_1$ such that

$$X_{t_\ell}(\alpha) \neq X_t(\alpha).$$

The alleged code for t requires a branch through

$$S^{X_t(\alpha)}_{\eta,d(\beta),\alpha}.$$

At this coordinate the unique intentional active code, namely the code for t_ℓ, uses the distinct tree

$$S^{X_{t_\ell}(\alpha)}_{\eta,d(\beta),\alpha}.$$

Hence the tree required by the alleged code is unused by the partial run. By Lemma 5.8, it remains Suslin after forcing with $\mathbb{P}$, contradicting the existence of the alleged branch. Therefore $t_\ell = t$, and the alleged code is one of the intentional codes. □

Remark 5.13. Finite reuse of a container is handled by different derived reservoir traces, not by spatially disjoint slots. The derived traces may have countable intersections, but they are pairwise almost disjoint. Therefore each intentional use has cofinally many clean blocks, and the no-unwanted-code argument can thin any alleged cofinal trace to one of these clean blocks. This is why the definition of SepCoded needs only positive branch information.

6 Clean allowability and the separation iteration

We now define the separation iteration. Its atomic coding blocks are the direct E-complete container codings introduced above, and the forcing is arranged to preserve CH. The activation and allowability terminology is a generalized Baire-space version of the allowable iterations used in the classical projective separation/reduction constructions [12, 14, 13]

6.1 Sealed containers

Definition 6.1 (Sealed containers). *During the main iteration we maintain an increasing sequence*

$$\langle B_\alpha : \alpha < \omega_2 \rangle$$

of subsets of ω_2. The set B_α consists of the containers used before stage α; its elements are sealed. A coding block inserted at stage α, and every coding block occurring inside an allowable forcing inserted at stage α, must use only containers outside B_α. If the stage uses the set E_α^{cont} of containers, then

$$B_{\alpha+1} = B_\alpha \cup E_\alpha^{\mathrm{cont}}.$$

At limit stages take unions.

Lemma 6.2. *For every $\alpha < \omega_2$, $|B_\alpha| < \omega_2$.*

Proof. Each stage inserts a forcing with a presentation in $H(\omega_2)$ and hence uses fewer than ω_2 many containers. Since ω_2 is regular, the union of $< \omega_2$ many such sets still has size $< \omega_2$. □

Definition 6.3 (Clean allowability). *A legal partial run is a countable-support iteration of direct coding blocks, together with the bookkeeping data for the containers and derived reservoir traces used by those blocks, which satisfies the syntactic requirements imposed on such blocks in Section 5. Let $B \subseteq \omega_2$. A legal partial run is B-*clean *if it avoids every container in B and uses each container only finitely many times, with pairwise distinct derived reservoir traces for the distinct uses of that container. A* clean iteration *means a countable-support legal partial run which is B-clean for the relevant sealed set B; when B is not displayed, it is the sealed set fixed by the surrounding construction. In the actual iteration, used containers are sealed, so a container is not reused by later stages.*

6.2 Allowability

Definition 6.4 (0-allowable). *A forcing is 0-allowable if it has a small presentation as a countable-support clean iteration of direct coding blocks, in the sense of Definition 6.3 with no external sealed set. Small means that the presentation belongs to $H(\omega_2)$.*

Lemma 6.5. *Every 0-allowable forcing is E-complete. If CH holds in the ambient model, every atomic direct coding block has size $\aleph_1$.*

Proof. Atomic direct coding blocks are E-complete by Lemma 5.3. Countable-support iterations of E-complete forcings are E-complete by Lemma 3.5. The size statement is part of Lemma 5.3. □

Suppose a pair (m, k, p) is considered at stage ρ and has not been neutralized. It becomes active at ρ if no current B_ρ-clean allowable forcing over $W[G_\rho]$ can force

$$\exists z \left(M_m(z, p) \wedge M_k(z, p) \right).$$

At activation we record the sealed set B_ρ and work from then on inside the subclass of allowable forcings avoiding B_ρ and obeying the side-placement rule for this pair. This set B_ρ is also the error term for the final separator: containers in B_ρ may contain arbitrary pre-activation branch noise and are ignored by the decoding relation used to define the separator. This sealed class is denoted

$$\Gamma^*_\rho(m, k, p).$$

Definition 6.6 (Side-placement rule)**.** *Let (m, k, p) activate at stage ρ. At a later stage $\beta \geq \rho$, if the bookkeeping presents x, then:*

(i) *if some B_β-clean member of $\Gamma^*_\rho(m, k, p)$ forces $M_m(x, p)$, the construction directly codes $(\rho, x, m, k, p, 0)$ into a fresh container outside B_β;*

(ii) *otherwise it directly codes $(\rho, x, m, k, p, 1)$ into a fresh container outside B_β.*

The side-one clause is a default placement.

Lemma 6.7. *The allowable classes, and the sealed classes $\Gamma^*_\rho(m, k, p)$, are closed under countable-support concatenation of legal clean partial runs.*

Proof. Concatenating presentations gives another legal presentation. Requirements already imposed on each piece are still obeyed in the concatenation. If both pieces avoid a sealed set B, then so does the concatenation. New names which appear only after the concatenation need not have been handled before they are presented by the bookkeeping; this is the usual partial-run convention. □

6.3 The global iteration

Fix a bookkeeping function

$$F : \omega_2 \to H(\omega_2)$$

which lists all relevant names for triples (x, p, m, k) unboundedly often. We build a countable-support iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$$

over W, simultaneously with the sealed-container sequence $\langle B_\alpha : \alpha < \omega_2 \rangle$. At stage α, if $F(\alpha)$ is not a well-formed name for a tuple (x, p, m, k), the iterand is trivial. Otherwise evaluate the tuple in $W[G_\alpha]$.

If the pair (m, k, p) has already been neutralized, do nothing. If the current model already satisfies

$$\exists z \left(M_m(z, p) \wedge M_k(z, p) \right),$$

declare the pair neutralized. If some current B_α-clean allowable forcing can force this intersection, insert the $<_L$-least such forcing and declare the pair neutralized. If no such forcing exists, activate the pair if it is not yet active. Let ρ be its activation stage. Then handle the present point x according to Definition 6.6, using a fresh container outside B_α and sealing it after use.

Lemma 6.8. *The final forcing $\mathbb{P}_{\omega_2}$ is E-complete, E-proper, adds no reals, and satisfies the ω_2-chain condition. Consequently the full extension preserves cardinals and satisfies*

$$2^\omega = \omega_1, \qquad 2^{\omega_1} = \omega_2.$$

Proof. Every atomic direct coding block is E-complete and, at every bounded stage, has size $\aleph_1$. The bounded-stage CH needed for the size computation is provided by Theorem 3.6, since the iterands are E-proper of size $\aleph_1$. The global iteration is E-complete by Lemma 3.5, hence adds no reals by Lemma 3.3. Therefore CH holds in the final model. The same Abraham–Shelah theorem, Theorem 3.6, gives the ω_2-c.c. for the countable-support iteration of length ω_2 of E-proper size-$\aleph_1$ iterands. Thus cardinals are preserved. The preliminary forcing added ω_2 many ω_1-Cohen subsets of ω_1, so $2^{\omega_1} \geq \omega_2$. The total forcing has size ω_2 and satisfies the ω_2-c.c.; using GCH in L and CH in the final model, the usual name counting gives $2^{\omega_1} \leq \omega_2$. □

Lemma 6.9. *Let $G \subseteq \mathbb{P}_{\omega_2}$ be generic over W. Every element of $(2^{\omega_1})^{W[G]}$ belongs to some bounded intermediate extension $W[G_\delta]$, $\delta < \omega_2$. Consequently, if $M_m(x,p)$ holds in the final model, then it already holds with witnesses in some bounded intermediate model.*

Proof. Let $\dot{x}$ be a name for an element of 2^{ω_1}. For each $\xi < \omega_1$, choose a maximal antichain deciding $\dot{x}(\xi)$. The final forcing is ω_2-c.c., so each antichain has size at most ω_1. There are ω_1 many coordinates, and every condition has countable support. Thus the union of the supports used by all these antichains has size at most ω_1, hence is bounded in ω_2. The name is therefore a $\mathbb{P}_\delta$-name for some $\delta < \omega_2$.

If $M_m(x,p)$ holds, choose the Σ^1_1 witness $y \in 2^{\omega_1}$ and apply the previous paragraph to x, p, y. Since the matrix θ_m is closed, its truth is absolute once the relevant objects are present. □

Lemma 6.10. *If (m,k,p) activates at stage ρ, then every later tail $\mathbb{P}_\beta/G_\rho$, $\rho < \beta \leq \omega_2$, is a B_ρ-clean legal partial run, in the sense of Definition 6.3, and belongs to the sealed class $\Gamma^*_\rho(m,k,p)$.*

Proof. After activation, all later stages obey the recorded side-placement requirement for the pair. All containers in B_ρ remain sealed forever, so later tails avoid them. Later requirements only shrink the class further. Hence the tail is a member of the sealed class attached at ρ. □

6.4 Relocation by Cohen-reservoir homogeneity

We next record the relocation principle used in the side-placement argument. The point is not that the branch-adding forcings attached to different Suslin-tree coordinates are literally isomorphic. What we use is weaker. The preliminary reservoir forcing contains ω_2 many ω_1-Cohen traces, and the existence of an allowable witness is already forced by a small condition in this homogeneous product. By permuting the reservoir coordinates we can make such a condition speak about fresh resources, and we can arrange that the shifted condition still belongs to the fixed reservoir generic. No definability of the individual reservoirs is used here; only the product homogeneity and the fact that all relevant names have support of size $< \omega_2$ are used. Let

$$\mathbb{C} = \mathrm{Add}_{\omega_1}(\omega_2)^L$$

be the countable-support product used in the preliminary construction to add the reservoir sequence

$$\vec{C} = \langle C_\nu : \nu < \omega_2 \rangle.$$

Let $G_C \subseteq \mathbb{C}$ denote the fixed reservoir generic. If $c \in \mathbb{C}$, write

$$\mathrm{supp}(c) = \{\nu < \omega_2 : (\exists \xi < \omega_1)\, c(\nu,\xi) \text{ is defined}\}.$$

Thus $\operatorname{supp}(c)$ is countable.

For a clean allowable forcing $\mathbb{Q}$, fix once and for all a small presentation of $\mathbb{Q}$ as a countable-support iteration of direct coding blocks. Let

$$\mathrm{Cont}(\mathbb{Q})$$

be the set of container indices occurring in this presentation, and let

$$\mathrm{Tr}(\mathbb{Q})$$

be the set of reservoir-trace indices occurring in it. Since the presentation belongs to $H(\omega_2)$, both sets have cardinality $< \omega_2$. Similarly, for the actual iteration, let

$$R_\alpha$$

denote the set of reservoir-trace indices used before stage α.

Lemma 6.11. *For every $\alpha < \omega_2$,*

$$|R_\alpha| < \omega_2.$$

Proof. At each stage the iterand has a small presentation and therefore uses fewer than ω_2 many derived reservoir traces. Since ω_2 is regular, the union of $< \omega_2$ many such sets still has cardinality $< \omega_2$. □

If π is a permutation of ω_2, let

$$\widehat{\pi} : \mathbb{C} \to \mathbb{C}$$

be the induced automorphism of the reservoir product:

$$\widehat{\pi}(c)(\pi(\nu), \xi) = c(\nu, \xi).$$

We extend $\widehat{\pi}$ recursively to $\mathbb{C}$-names in the usual way.

Lemma 6.12. *Let $c \in \mathbb{C}$, and let*

$$A_0, A_1, R \subseteq \omega_2$$

have cardinality $< \omega_2$, with $A_0 \cap A_1 = \emptyset$. Think of A_0 as the set of reservoir coordinates which must be fixed, A_1 as the set of coordinates supporting a witness, and R as the set of forbidden coordinates. Then below c it is dense to find a condition q and a permutation π of ω_2 such that

$$\pi \restriction A_0 = \mathrm{id}, \qquad \pi[A_1] \cap R = \emptyset,$$

and

$$q \leq c, \qquad q \leq \widehat{\pi}(c).$$

Consequently, if $c \in G_C$, then there is such a permutation π with

$$\widehat{\pi}(c) \in G_C.$$

Proof. Let $r \leq c$ be arbitrary. Since

$$|A_0 \cup A_1 \cup R \cup \operatorname{supp}(r)| < \omega_2$$

and ω_2 is regular, we may choose a permutation π of ω_2 which fixes A_0 pointwise and sends

$$A_1 \cup (\operatorname{supp}(c) \setminus A_0)$$

away from

$$R \cup \operatorname{supp}(r)$$

except for the part of $\operatorname{supp}(c)$ already lying in A_0. This is possible because all mentioned sets have size $< \omega_2$.

The conditions r and $\widehat{\pi}(c)$ are compatible. On coordinates in A_0, the condition $\widehat{\pi}(c)$ agrees with c, and $r \leq c$. On coordinates outside A_0, the support of $\widehat{\pi}(c)$ was moved away from $\operatorname{supp}(r)$. Let q be a common extension of r and $\widehat{\pi}(c)$. Then $q \leq r \leq c$ and $q \leq \widehat{\pi}(c)$. This proves density below c. If $c \in G_C$, the generic filter meets this dense set below c. Thus for some $q \in G_C$ and some such π, we have $q \leq \widehat{\pi}(c)$. Since filters are upward closed, $\widehat{\pi}(c) \in G_C$. □

The next lemma is the exact relocation statement used below. It deliberately asserts only the existence of a fresh allowable witness. It does not assert that an already chosen branch-adding forcing is isomorphic to its relocated copy. The "support" of a witness refers to the reservoir coordinates on which the small presentation of the witness, including its auxiliary choices of traces and containers, depends.

Lemma 6.13. *Work in an intermediate model containing the reservoir generic G_C. Let*

$$B, R \subseteq \omega_2$$

be forbidden sets of container and reservoir-trace indices, both of cardinality $< \omega_2$. Suppose that $x, p_0 \in 2^{\omega_1}$ have reservoir support contained in a set $A_0 \subseteq \omega_2$ of cardinality $< \omega_2$. Suppose further that for some $c \in G_C$ and some $A_1 \subseteq \omega_2$ of cardinality $< \omega_2$, disjoint from A_0, c forces over the reservoir product that there is a clean sufficiently allowable forcing whose auxiliary resources are supported by A_1 and which forces

$$M_m(x, p_0).$$

Then, in the actual reservoir extension, there is a clean sufficiently allowable forcing $\mathbb{Q}^*$ which uses no container from B, uses no reservoir trace from R, and satisfies

$$\mathbb{Q}^* \Vdash M_m(x, p_0).$$

Proof. Apply Lemma 6.12 with the forbidden set enlarged to include R and the reservoir supports of the names which decide the container choices of the small presentation. Since B has cardinality $< \omega_2$, the permutation can also be chosen so that the shifted container choices avoid B. We obtain a permutation π of ω_2 such that

$$\widehat{\pi}(c) \in G_C, \qquad \pi \restriction A_0 = \mathrm{id}, \qquad \pi[A_1] \cap R = \emptyset,$$

and such that the container choices in the shifted presentation avoid B. Since $\widehat{\pi}$ is an automorphism of the reservoir product, $\widehat{\pi}(c)$ forces the π-shift of the statement forced by c. The parameters x and p_0 are fixed because their reservoir support is contained in A_0, and π fixes A_0 pointwise. Hence $\widehat{\pi}(c)$ forces that there is a clean sufficiently allowable forcing, using the shifted auxiliary resources, which forces $M_m(x, p_0)$. These shifted resources avoid B and R by construction.

Since $\widehat{\pi}(c) \in G_C$, the actual reservoir extension contains such a witness. Call it $\mathbb{Q}^*$. Then $\mathbb{Q}^*$ is clean over the forbidden resources and

$$\mathbb{Q}^* \Vdash M_m(x, p_0).$$

□

Lemma 6.14. *Let $\mathbb{R}$ be a clean member of a sealed class $\Gamma^*_\rho(m,k,p)$, and let $r \in \mathbb{R}$. Then the restriction*

$$\mathbb{R} \upharpoonright r = \{s \in \mathbb{R} : s \leq r\}$$

is again a clean member of $\Gamma^*_\rho(m,k,p)$.

Proof. The restriction uses the same presentation and the same resources as $\mathbb{R}$. It therefore avoids the same sealed containers, uses each container only finitely often, and preserves all side-placement requirements. □

Lemma 6.15. *If $M_m(x,p)$ holds in an intermediate extension, then it remains true in every further forcing extension preserving ω_1.*

Proof. Write $M_m(x,p)$ as a Σ^1_1 assertion

$$\exists y\, \theta_m(x,p,y),$$

where θ_m is closed. If y witnesses the assertion in the intermediate model, then the same y remains present in every further extension. Since closed membership is absolute once ω_1 is preserved, $\theta_m(x,p,y)$ remains true. □

7 Correctness of the construction

Lemma 7.1. *If a pair (m,k,p) is neutralized at some stage, then in every later stage*

$$\exists z\, \big(M_m(z,p) \wedge M_k(z,p)\big)$$

holds.

Proof. The intersection statement is a Σ^1_1 assertion with actual witnesses $z, y_0, y_1 \in 2^{\omega_1}$. Once these witnesses exist, they remain witnesses in every later forcing extension. □

Lemma 7.2. *Suppose (m,k,p) activates at stage ρ. If at a later stage the construction codes*

$$(\rho, x, m, k, p, 0),$$

then $M_k(x,p)$ is false in the final model. If at a later stage the construction codes

$$(\rho, x, m, k, p, 1),$$

then $M_m(x,p)$ is false in the final model.

Proof. We first prove the side-zero case. Suppose that the construction codes

$$(\rho, x, m, k, p, 0)$$

at stage $\beta \geq \rho$. By the definition of side zero, in $W[G_\beta]$ there is a B_β-clean member

$$\mathbb{Q} \in \Gamma^*_\rho(m,k,p)$$

such that

$$\mathbb{Q} \Vdash M_m(x,p).$$

Assume toward a contradiction that $M_k(x,p)$ holds in the final model. By Lemma 6.9, there is some $\gamma > \beta$ such that the relevant witnesses already appear in $W[G_\gamma]$. By the forcing theorem, strengthening if necessary inside the actual quotient generic, there is a condition

$$r \in \mathbb{P}_\gamma / G_\beta$$

such that

$$r \Vdash_{\mathbb{P}_\gamma / G_\beta} M_k(x, p).$$

Let

$$\mathbb{R} = (\mathbb{P}_\gamma / G_\beta) \restriction r.$$

By Lemma 6.10 and Lemma 6.14, $\mathbb{R}$ is a clean member of the sealed class $\Gamma^*_\rho(m, k, p)$ over $W[G_\beta]$. Now work over the activation model $W[G_\rho]$. Consider first the initial tail

$$\mathbb{P}_\beta / G_\rho,$$

and then the restricted tail $\mathbb{R}$. In the extension by these two factors, the sets of containers and derived reservoir traces used before stage γ are still of size $< \omega_2$. Choose reservoir supports for the parameters x, p and a reservoir condition in the fixed generic witnessing the statement that the side-zero forcing exists. Applying Lemma 6.13 with

$$B = B_\gamma, \qquad R = R_\gamma,$$

we obtain a fresh

$$\mathbb{Q}^* \in \Gamma^*_\rho(m, k, p)$$

which avoids all containers and derived reservoir traces used before stage γ and satisfies

$$\mathbb{Q}^* \Vdash M_m(x, p).$$

Thus the three-step forcing

$$(\mathbb{P}_\beta / G_\rho) * \mathbb{R} * \mathbb{Q}^*$$

is a clean member of $\Gamma^*_\rho(m, k, p)$, by Lemma 6.7. The middle factor $\mathbb{R}$ forces $M_k(x, p)$. By Lemma 6.15, this remains true after the further forcing with $\mathbb{Q}^*$. The final factor $\mathbb{Q}^*$ forces $M_m(x, p)$. Therefore the three-step forcing forces

$$M_m(x, p) \wedge M_k(x, p).$$

In particular, it forces

$$\exists z \left(M_m(z, p) \wedge M_k(z, p) \right),$$

with $z = x$. This contradicts the activation of (m, k, p) at stage ρ, since activation says precisely that no clean member of $\Gamma^*_\rho(m, k, p)$ over $W[G_\rho]$ can force such an intersection. Hence $M_k(x, p)$ is false in the final model.

We now prove the side-one case. Suppose that the construction codes

$$(\rho, x, m, k, p, 1)$$

at stage $\beta \geq \rho$. By the definition of the default side-one placement, in $W[G_\beta]$ there is no B_β-clean member of $\Gamma^*_\rho(m, k, p)$ forcing $M_m(x, p)$. Assume toward a contradiction that $M_m(x, p)$ holds in the final model. Again by Lemma 6.9 and the forcing theorem, there are $\gamma > \beta$ and

$$r \in \mathbb{P}_\gamma / G_\beta$$

such that

$$r \Vdash_{\mathbb{P}_\gamma / G_\beta} M_m(x, p).$$

Let

$$\mathbb{R} = (\mathbb{P}_\gamma / G_\beta) \restriction r.$$

By Lemma 6.10 and Lemma 6.14, $\mathbb{R}$ is a B_β-clean member of $\Gamma^*_\rho(m,k,p)$. But $\mathbb{R}$ forces $M_m(x,p)$, contrary to the side-one choice made at stage β. Therefore $M_m(x,p)$ is false in the final model. □

Let (m,k,p) be a pair which is not neutralized in the final model, and let ρ be its activation stage. The activation-stage sealed set B_ρ is the error term. Define

$$D_{m,k,p} = \{x \in 2^{\omega_1} : \mathrm{SepCoded}^{B_\rho}_\rho(x,m,k,p,0)\}.$$

Thus the separator ignores all containers which had already been used before the pair activated.

Lemma 7.3 (Total side assignment)**.** *For every $x \in 2^{\omega_1}$ in the final model, exactly one of*

$$\mathrm{SepCoded}^{B_\rho}_\rho(x,m,k,p,0), \qquad \mathrm{SepCoded}^{B_\rho}_\rho(x,m,k,p,1)$$

holds.

Proof. By bounded appearance, x and p occur in some bounded intermediate model after activation. At a later bookkeeping occurrence of (x,p,m,k), the construction assigns one of the two sides and directly codes the corresponding tuple into a fresh container outside the current sealed set, hence outside B_ρ. Thus at least one clean side code exists, i.e., at least one of the two displayed $\mathrm{SepCoded}^{B_\rho}_\rho$ relations holds. By the no-unwanted-code lemma, Lemma 5.12, every counted clean code is intentional. The construction records once a point has been assigned a side for the activated pair and does nothing at later occurrences of the same point. Hence two opposite intentional clean side codes for the same x cannot be produced. □

Lemma 7.4. *The set $D_{m,k,p}$ is Δ^1_1.*

Proof. The positive definition is the Σ^1_1 relation $\mathrm{SepCoded}^{B_\rho}_\rho(x,m,k,p,0)$, with B_ρ used as a boldface parameter. By total side assignment, the complement is defined by the Σ^1_1 relation $\mathrm{SepCoded}^{B_\rho}_\rho(x,m,k,p,1)$. Hence $D_{m,k,p}$ is both Σ^1_1 and Π^1_1. □

Lemma 7.5. *If (m,k,p) is not neutralized and $M_m(x,p)$ holds in the final model, then $x \in D_{m,k,p}$.*

Proof. By Lemma 6.9, the witnesses to $M_m(x,p)$ occur in some bounded intermediate model. At a later bookkeeping occurrence, the trivial forcing is a clean member of the sealed class which forces $M_m(x,p)$. Therefore the construction assigns side zero to x and writes the code into a fresh container outside the current sealed set, in particular outside B_ρ. □

Lemma 7.6. *If (m,k,p) is not neutralized, then*

$$D_{m,k,p} \cap \{x : M_k(x,p)\} = \emptyset.$$

Proof. If $x \in D_{m,k,p}$, then some container outside the error term B_ρ witnesses $\mathrm{SepCoded}^{B_\rho}_\rho(x,m,k,p,0)$. By Lemma 5.12, this side-zero code is intentional. Since containers outside B_ρ were not used before activation, the intentional code was produced after the pair activated. Lemma 7.2 says that such a point never enters the k-side. □

Theorem 7.7 (Main theorem). *In the final forcing extension, every two disjoint boldface* $\mathbf{\Sigma}^1_1$ *subsets of* 2^{ω_1} *are separated by a boldface* $\mathbf{\Delta}^1_1$ *set. Moreover*

$$2^\omega = \omega_1, \qquad 2^{\omega_1} = \omega_2.$$

Proof. The cardinal arithmetic was proved in Lemma 6.8. Let the two disjoint Σ^1_1 sets be defined by $M_m(x,p)$ and $M_k(x,p)$. If the pair (m,k,p) were neutralized, Lemma 7.1 would give an intersection in the final model. Hence the pair is active at some stage ρ. Form $D_{m,k,p}$ as above. By Lemma 7.5, the m-side is contained in $D_{m,k,p}$. By Lemma 7.6, $D_{m,k,p}$ is disjoint from the k-side. By Lemma 7.4, $D_{m,k,p}$ is Δ^1_1. □

8 Open questions

Question 1. *Can the direct* ω_1*-length coding mechanism be strengthened to handle reduction, not only separation?*

Question 2. *Does the separation model have consequences for the Borel* versus* Δ^1_1 *problem beyond the separation consequence proved here?*

Question 3. *Can the same direct-code architecture be lifted to regular* $\kappa > \omega_1$ *satisfying* $\kappa^{<\kappa} = \kappa$*?*

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