

A short skeleton for the lifted Fischer–Törnquist maximal orthogonal family construction

Abstract

This note isolates the part of the local Delfino construction which is needed for the maximal-orthogonal-family refinement. We record only the black-box consequences of the coding-over- W_2 machinery and then explain how the Fischer–Törnquist coding trick yields a Π_n^1 maximal orthogonal family, while the regularity and uniformization part of the construction rules out boldface Σ_n^1 maximal orthogonal families.

1 Introduction

Fix $n \geq 3$ and assume that the canonical mouse M_{n-2} exists. The local Delfino construction over M_{n-2} produces a convenient generic extension W_2 with a flexible projective coding predicate. The point of the present note is that this coding predicate can be used not only for the Δ_{n+1}^1 wellorder and for the Σ_{n+1+m}^1 uniformization scheme, but also for a maximal orthogonal family of measures.

The desired final constellation is:

$$\begin{aligned} \text{MA,} \quad & \text{all } \Sigma_n^1 \text{ sets have LM and BP,} \quad \Sigma_{n+1+m}^1\text{-uniformization for all } m < \omega, \\ & \Delta_{n+1}^1\text{-wellorder of the reals,} \quad \Pi_n^1\text{-definable mof,} \quad \text{no } \Sigma_n^1\text{-definable mof.} \end{aligned}$$

Here “mof” means a maximal family of pairwise orthogonal Borel probability measures on 2^ω . The negative half is the already existing regularity/uniformization argument. The positive half is a lifted version of Fischer–Törnquist and Fischer–Friedman–Törnquist: when a measure is accepted into the growing family, we code the relevant projective witness into an equivalent measure. Thus membership in the final family is checked by a Π_n^1 predicate because the apparent existential witness has become recursively readable from the measure itself.

This is only a skeleton. The forcing facts about W_2 are used as black boxes from the local Delfino construction.

2 The preparatory coding extension up to W_2

We use n for the level of the desired separation. Thus the base model is M_{n-2} . For $n = 3$ this is the M_1 construction; for larger n the same record is read with the corresponding Martin–Solovay trees and the corresponding Steel correctness facts.

2.1 The first preparatory universe W_0

Starting from M_{n-2} , perform the preparatory proper/Suslin-tree-preserving iteration used in the local Delfino construction. Denote the resulting model by

$$W_0.$$

We shall only use the following consequences.

Black box 2.1. *In W_0 :*

(W0.1) ω_1 is preserved and the relevant inaccessible of M_{n-2} has become ω_2 .

(W0.2) There is a canonical wellorder of $P(\omega_1)^{W_0}$.

(W0.3) There is a uniformly definable sequence

$$\vec{T} = \langle T_\xi : \xi < \omega_2 \rangle$$

of independent Suslin trees.

(W0.4) *In every ccc extension of W_0 , the old $H(\omega_2)^{W_0}$, the wellorder of $P(\omega_1)^{W_0}$, and the sequence \vec{T} are still recognizable.*

(W0.5) *There are cofinally many suitable \aleph_1 -sized models below ω_2 ; suitable models compute the relevant initial pieces of \vec{T} correctly.*

The role of W_0 is only to provide a robust stock of definable Suslin trees and suitable models. The later coding predicates will be read inside these suitable models.

2.2 The intermediate universe W_1

Let U be the set of reals coding suitable models, relative to the fixed almost-disjoint coding apparatus. Force over W_0 with an almost-disjoint coding forcing

$$\mathbb{A}(U)$$

which adds a real r_U satisfying, for every old real $x \in W_0$,

$$x \in U \iff r_U \cap S(x) \text{ is finite.}$$

Next write r_U into the first ω -block \vec{T}^0 of the Suslin-tree sequence. Namely, define a finite-support product

$$\mathbb{D}(r_U) = \prod_{m < \omega}^{\text{fin}} \mathbb{R}_m,$$

where

$$\mathbb{R}_m = \begin{cases} T_m^0, & r_U(m) = 1, \\ \text{Sp}(T_m^0), & r_U(m) = 0. \end{cases}$$

Here T_m^0 is used as the forcing which adds a cofinal branch, and $\text{Sp}(T_m^0)$ is Baumgartner's specializing forcing. By independence of the T_m^0 's, this finite-support product is ccc. Let

$$W_1 = W_0[H_U][H_D] := W_0^{\mathbb{A}(U) * \mathbb{D}(r_U)}.$$

Black box 2.2. *The model W_1 is a ccc extension of W_0 and has the following property. In every outer ccc extension of W_1 there is a $\Sigma_1(\vec{C}, \vec{T}^0)$ definition which recognizes reals coding suitable models. Consequently suitable models can correctly reconstruct the old sequence \vec{T} and can read the branch/specialization patterns written into \vec{T} .*

This is the first “coding over codes” step: the real r_U says which reals code suitable models, and r_U itself is written into a definable Suslin-tree block.

2.3 The final preparatory universe W_2

The model W_2 is designed so that the later iteration can simultaneously force MA and use fresh Suslin-tree blocks for projective coding without losing control of the unused tail.

First force over W_1 with the finite-support product of Cohen forcing

$$\mathbb{P}_0 = \prod_{\alpha < \omega_2}^{\text{fin}} \mathbb{C}.$$

Let c_α be the α -th Cohen real. By the Shelah–Todorćević construction, each c_α defines a Suslin tree S_α . Put

$$\vec{S} = \langle S_\alpha : \alpha < \omega_2 \rangle.$$

Finite products of the S_α ’s remain Suslin; hence \vec{S} is an independent sequence.

Second, code each S_α into the old sequence \vec{T} . For $\alpha < \omega_2$ and $\beta < \omega_1$, set

$$\mathbb{P}_{1,\alpha}^\beta = \begin{cases} T_{\omega_1\alpha+\beta}, & S_\alpha(\beta) = 1, \\ \text{Sp}(T_{\omega_1\alpha+\beta}), & S_\alpha(\beta) = 0. \end{cases}$$

Let

$$\mathbb{P}_{1,\alpha} = \prod_{\beta < \omega_1}^{\text{fin}} \mathbb{P}_{1,\alpha}^\beta, \quad \mathbb{P}_1 = \prod_{\alpha < \omega_2}^{\text{fin}} \mathbb{P}_{1,\alpha},$$

and define

$$W_2 = W_1^{\mathbb{P}_0 * \mathbb{P}_1}.$$

Black box 2.3. *The universe W_2 satisfies:*

(W2.1) *W_2 is a cardinal-preserving ccc extension of W_0 and, after the harmless normalization used in the local Delfino construction,*

$$W_2 \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2.$$

(W2.2) *The sequence \vec{S} is uniformly $\Sigma_1(\omega_1)$ -definable over W_2 .*

(W2.3) *We write $\vec{S}^\xi = \langle S_m^\xi : m < \omega \rangle$ for the ξ -th ω -block of \vec{S} . The blocks form an independent sequence of independent Suslin-tree blocks.*

(W2.4) *If $\mathbb{Q} \in W_2$ is ccc of size \aleph_1 , then \mathbb{Q} belongs to some initial part $W_1[H_\gamma]$ of the construction of W_2 . The tail blocks \vec{S}^ξ for $\xi > \gamma$ remain independent Suslin-tree blocks after forcing with \mathbb{Q} .*

The last item is the reason for passing from W_1 to W_2 . It is the tail preservation feature which allows the final iteration to include ordinary ccc forcing for MA while reserving later blocks of \vec{S} for intentional projective codes.

3 The flexible Σ_{n+1}^1 coding predicate

Over W_2 one defines a coding forcing using a fresh block \vec{S}^ξ . For a real z , the first part of the forcing writes the bits of z into the branch/specialization pattern on \vec{S}^ξ :

$$\mathbb{D}_\xi(z) = \prod_{m < \omega}^{\text{fin}} \mathbb{R}_m^{\xi, z}, \quad \mathbb{R}_m^{\xi, z} = \begin{cases} S_m^\xi, & z(m) = 1, \\ \text{Sp}(S_m^\xi), & z(m) = 0. \end{cases}$$

The second part is the almost-disjoint coding step which adds a real coding the branches, specializing functions, and suitable-model data needed to verify the pattern in countable suitable models. We denote the resulting forcing by

$$\text{Code}_n(z, \xi).$$

The subscript n records the projective level of the intended separation.

Black box 3.1 (surgical coding). *There is a formula*

$$\sigma_n(z) \iff \exists r \psi_n(z, r)$$

such that:

- (C1) ψ_n is Π_n^1 , hence σ_n is Σ_{n+1}^1 .
- (C2) In W_2 , no real satisfies σ_n .
- (C3) For every real $z \in W_2$ and every unused block $\xi < \omega_2$, the forcing $\text{Code}_n(z, \xi)$ is ccc of size \aleph_1 and forces $\psi_n(z, r_z)$ for a canonical real r_z added by the coding construction.
- (C4) The forcing $\text{Code}_n(z, \xi)$ does not make $\sigma_n(z')$ true for any unintended real $z' \neq z$ in the same coding namespace.
- (C5) The assertion is persistent under further ccc forcing which uses only fresh blocks of \vec{S} .

For $n = 3$ this is the Σ_4^1 predicate explicitly built in the M_1 version of the local Delfino paper. For general n , the same definition is lifted over M_{n-2} by replacing the M_1 correctness input by the corresponding M_{n-2} Martin–Solovay correctness input. The important point here is not the internal definition of ψ_n , but its two external features: it is Π_n^1 , and its existential closure can be changed surgically by ccc forcing on one fresh \vec{S} -block.

4 The final iteration over W_2

Let

$$\langle (\mathbb{P}_\beta, \dot{Q}_\beta) : \beta < \omega_2 \rangle$$

be a finite-support ccc iteration over W_2 . The bookkeeping is partitioned into cofinal classes. The old classes are:

- (1) diagonal MA stages;
- (2) wellorder coding stages;

(3) global Σ -uniformization stages.

For the present refinement we add a fourth cofinal class:

(4) maximal-orthogonal-family stages.

At every coding stage the least still unused block of \vec{S} is consumed. Thus the used blocks form an initial segment and the unused tail remains independent.

At a diagonal MA stage the bookkeeping presents a ccc forcing \mathbb{B} of size \aleph_1 , together with its dense sets. If \mathbb{B} already belongs to a sufficiently early initial part of the construction of W_2 , we force with \mathbb{B} ; otherwise we force trivially. By Black box 2.3, this does not damage the unused tail of \vec{S} .

At the wellorder and uniformization stages one uses the forcings $\text{Code}_n(z, \xi)$ and their higher iterates, exactly as in the local Delfino construction. These stages yield the Δ_{n+1}^1 wellorder and the Σ_{n+1+m}^1 uniformization scheme. The maximal-orthogonal-family stages use the same coding forcings but with a separate tag.

Let $G_{\omega_2} \subseteq \mathbb{P}_{\omega_2}$ be generic and put

$$V^* = W_2[G_{\omega_2}].$$

Black box 4.1. *The old part of the construction gives, in V^* :*

(F1) MA holds.

(F2) Every boldface Σ_n^1 set of reals is Lebesgue measurable and has the Baire property.

(F3) There is a Δ_{n+1}^1 wellorder of the reals.

(F4) Σ_{n+1+m}^1 -uniformization holds for every $m < \omega$.

Adding the mof stages below does not affect these conclusions, since those stages are ccc forcings of size \aleph_1 of the same fresh-block coding form.

5 The Fischer–Törnquist measure-coding lemma

We use the standard recursive coding of Borel probability measures on 2^ω . Let $p(2^\omega)$ be the recursive Polish space of measure codes, and let

$$p_c(2^\omega) \subseteq p(2^\omega)$$

be the arithmetical set of codes for non-atomic measures. If $f \in p(2^\omega)$, write μ_f for the corresponding measure. The relations

$$\mu_f \perp \mu_g, \quad \mu_f \approx \mu_g, \quad f \in p_c(2^\omega)$$

are arithmetical in f and g .

Lemma 5.1 (Fischer–Törnquist). *There is a recursive partial decoding map r_{FT} on measure codes and a recursive function*

$$G : p_c(2^\omega) \times 2^\omega \longrightarrow p_c(2^\omega)$$

such that for every $f \in p_c(2^\omega)$ and every real u ,

$$\mu_{G(f,u)} \approx \mu_f \quad \text{and} \quad r_{\text{FT}}(G(f,u)) = u.$$

Sketch. One recursively chooses splitting nodes for the non-atomic measure μ_f and changes the conditional weights at those nodes to $1/3$ and $2/3$ according to the bits of u . The resulting measure is absolutely equivalent to μ_f , while the pattern of ratios recursively recovers u from the new code. No other feature of the construction is used below. \square

6 The mof stages

We now define the new bookkeeping class. Along the iteration we construct a family

$$\mathcal{O}_{\text{na}} \subseteq P_c(2^\omega)$$

of non-atomic measures.

Definition 6.1 (mof stage). *Suppose that at stage β the bookkeeping presents a \mathbb{P}_β -name \dot{f} for a real. In $W_2[G_\beta]$, let*

$$f = \dot{f}^{G_\beta}.$$

If $f \notin p_c(2^\omega)$, or if μ_f is not orthogonal to the family already constructed, force trivially and add no measure.

If $f \in p_c(2^\omega)$ and μ_f is orthogonal to the family already constructed, choose the least unused block ξ_β of \vec{S} and force with

$$\text{Code}_n(\langle \text{mof}, f \rangle, \xi_\beta).$$

Let r_β be the canonical real witnessing

$$\psi_n(\langle \text{mof}, f \rangle, r_\beta)$$

in the extension by this coding forcing. Put

$$u_\beta = f \oplus r_\beta, \quad g_\beta = G(f, u_\beta),$$

and add μ_{g_β} to \mathcal{O}_{na} .

The tag **mof** is reserved for these stages only. It is not used by the wellorder or uniformization part of the construction.

Lemma 6.2. *In V^* , the family \mathcal{O}_{na} is a maximal orthogonal family of non-atomic measures.*

Sketch. Pairwise orthogonality is immediate by induction. If f is accepted at stage β , then μ_f is orthogonal to all earlier members. Since

$$\mu_{g_\beta} = \mu_{G(f, u_\beta)} \approx \mu_f,$$

the new measure is also orthogonal to all earlier members. Later stages test orthogonality against the enlarged family, so later accepted measures are orthogonal to μ_{g_β} as well.

For maximality, let $h \in p_c(2^\omega)$ code a non-atomic measure orthogonal to every member of the final family. Choose a stage after h appears at which the bookkeeping presents a name for h . At that stage h is still orthogonal to the current family, hence it is accepted. The construction adds $G(h, u)$ for some real u , and this code represents a measure equivalent to μ_h . This contradicts the assumption that μ_h is orthogonal to the final family. \square

7 The Π_n^1 definition

The definability calculation is the whole point of the Fischer–Törnquist trick. The witness r_β to the Σ_{n+1}^1 coding statement is not left outside the measure. It is inserted into the measure code itself.

For $g \in p_c(2^\omega)$, if $g \in \text{dom}(r_{\text{FT}})$, write

$$r_{\text{FT}}(g) = u_g, \quad u_g = f_g \oplus r_g$$

where f_g and r_g are the recursive projections of u_g .

Lemma 7.1. *The non-atomic family \mathcal{O}_{na} is Π_n^1 -definable.*

Sketch. Define $\mu_g \in \mathcal{O}_{\text{na}}$ by the following condition:

$$g \in p_c(2^\omega), \quad g \in \text{dom}(r_{\text{FT}}), \quad f_g \in p_c(2^\omega), \quad g = G(f_g, f_g \oplus r_g), \quad \psi_n(\langle \text{mof}, f_g \rangle, r_g). \quad (*)$$

All clauses except the last are arithmetical. The last clause is Π_n^1 by Black box 3.1. Therefore $(*)$ is a Π_n^1 condition.

If $g = g_\beta$ was added at a mof stage, then $r_{\text{FT}}(g) = f \oplus r_\beta$, $g = G(f, f \oplus r_\beta)$, and $\psi_n(\langle \text{mof}, f \rangle, r_\beta)$ holds. Hence $(*)$ holds.

Conversely, suppose $(*)$ holds. Then the no-accidental-coding part of Black box 3.1 implies that the code for $\langle \text{mof}, f_g \rangle$ was intentionally produced in the block named by the witness r_g . Since the tag mof is reserved for accepting mof stages, this happened at a mof stage, and the measure added there is exactly $\mu_{G(f_g, f_g \oplus r_g)} = \mu_g$. \square

Finally set

$$\mathcal{O} = \mathcal{O}_{\text{na}} \cup \{\delta_x : x \in 2^\omega\},$$

where δ_x is the point mass at x . The set of point measures is Borel, every point measure is orthogonal to every non-atomic measure, and distinct point measures are orthogonal. Thus \mathcal{O} is a Π_n^1 maximal orthogonal family of all Borel probability measures on 2^ω .

8 The absence of Σ_n^1 mofs

We also recall the negative argument in the form needed here.

Proposition 8.1. *Assume that every boldface Σ_n^1 set of reals has the Baire property and that every total boldface Σ_n^1 relation on the reals has a boldface Σ_n^1 uniformization. Then there is no boldface Σ_n^1 maximal orthogonal family of Borel probability measures on 2^ω .*

Sketch. Suppose $A \subseteq P(2^\omega)$ is a $\Sigma_n^1(a)$ maximal orthogonal family. Use the Kechris–Sofronidis product-measure map

$$x \mapsto \mu^x$$

for the turbulent equivalence relation E_I on 2^ω , so that

$$xE_I y \Rightarrow \mu^x \approx \mu^y, \quad \neg xE_I y \Rightarrow \mu^x \perp \mu^y.$$

For each x , the set of members of A not orthogonal to μ^x is countable, by the ccc-below property for absolute continuity. Let $Q(x, (\nu_i)_{i < \omega})$ say that $(\nu_i)_{i < \omega}$ enumerates exactly this countable

set. Since orthogonality and absolute continuity are arithmetical on measure codes, Q is a $\Sigma_n^1(a)$ relation with nonempty sections.

By Σ_n^1 uniformization, choose a $\Sigma_n^1(a)$ function F selecting such a sequence for every x . Since all $\Sigma_n^1(a)$ sets have the Baire property, F is Baire measurable. The assignment

$$x \mapsto \{F(x)(i) : i < \omega\}$$

is E_I -invariant. Hjorth turbulence for E_I implies that this assignment is constant on a comeager set. This contradicts the fact that all E_I -classes are meager and that non- E_I -equivalent reals yield orthogonal product measures. Hence no such A exists. \square

9 Conclusion

Combining the black boxes from the local Delfino construction with the mof stages gives the following final statement.

Theorem 9.1. *Let $n \geq 3$ and assume that M_{n-2} exists. There is a generic extension V^* of M_{n-2} such that:*

- (1) $V^* \models \text{MA}$ and $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$;
- (2) every boldface Σ_n^1 set of reals is Lebesgue measurable and has the Baire property;
- (3) there is a Δ_{n+1}^1 wellorder of the reals;
- (4) Σ_{n+1+m}^1 -uniformization holds for every $m < \omega$;
- (5) there is a Π_n^1 -definable maximal orthogonal family of Borel probability measures on 2^ω ;
- (6) there is no Σ_n^1 -definable maximal orthogonal family of Borel probability measures on 2^ω .

Proof sketch. Run the local Delfino construction over M_{n-2} up to W_2 . Over W_2 , perform the final finite-support ccc iteration with four cofinal bookkeeping classes: diagonal MA stages, wellorder stages, global Σ -uniformization stages, and the mof stages of Definition 6.1.

The old stages give MA, the Δ_{n+1}^1 wellorder, the Σ_{n+1+m}^1 uniformization scheme, and the regularity of all Σ_n^1 sets. The new mof stages are harmless for those proofs because they are the same kind of ccc fresh-block coding forcings already used by the construction. Lemmas 6.2 and 7.1 give the Π_n^1 maximal orthogonal family. Proposition 8.1 gives the nonexistence of Σ_n^1 maximal orthogonal families. \square

Remark 9.2. *For $n = 2$ the analogous phenomenon is the Fischer–Friedman–Törnquist model: a Π_2^1 mof exists, while no Σ_2^1 mof exists. The point of the present skeleton is that the W_2 coding mechanism supplies the higher mutable Σ_{n+1}^1 predicate whose Π_n^1 matrix can be inserted into the Fischer–Törnquist measure code.*

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