On a local variant of the 12th Delfino problem — the Π-side

Stefan Hoffelner[1] *

[1]TU Wien

June 11, 2026

Abstract

Assume that M_n, the canonical inner model with n Woodin cardinals, exists. We force a model with continuum $\aleph_2$ in which every $\boldsymbol{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable and has the Baire property, the Σ^1_{n+2}- and Π^1_{n+3}-uniformization properties hold, and the reals admit a Δ^1_{n+3}-definable well-order. Thus regularity up to a fixed finite projective level, together with a definable well-order of the reals at the adjacent level, does not force the determinacy strength which would normally explain that regularity, even when this package is strengthened by adjacent Σ- and Π-uniformization. In particular, this gives a negative answer to a local form of Woodin's twelfth Delfino problem asked by Friedman–Schindler.

1 Introduction

A central theme in descriptive set theory is the interaction between regularity properties, such as Lebesgue measurability and the Baire property, and definable choice principles for projective sets of reals. This paper concerns finite projective configurations in which regularity, uniformization, and definable well-ordering occur together. The canonical source of such configurations is the hierarchy of mice with finitely many Woodin cardinals. Let M_k denote the canonical inner model with k Woodin cardinals. Steel's analysis shows that M_k has a Δ^1_{k+2}-definable well-order of its reals [13, 12]. On the other hand, the Woodin cardinals in these models account, through the corresponding local determinacy and scale analysis, for regularity and uniformization phenomena at lower projective levels.

---

*The author's research was funded in whole by the Austrian Science Fund (FWF) Grant-DOI 10.55776/P37228. For the purpose of open access, the author has applied a CC BY public copyright license to any Author Accepted Manuscript version arising from this submission.

Thus M_{n+1} suggests a specific finite constellation in the projective hierarchy. Below the well-ordering level one has the regularity and scale-theoretic consequences associated with the local determinacy fragment; at the next level the fine structure of M_{n+1} supplies a Δ^1_{n+3}-definable well-order of the reals. The visible pattern relevant here is therefore regularity for boldface $\boldsymbol{\Sigma}^1_{n+2}$ sets, the corresponding finite uniformization behavior, and a Δ^1_{n+3}-definable well-order.

Before the present work, the systematic examples exhibiting this exact constellation came from M_{n+1} and closely related canonical models; in particular, they satisfied the corresponding $\boldsymbol{\Delta}^1_{n+1}$-determinacy fragment. Since the standard source for this behavior of the projective sets of reals is the M_{n+1}-picture, it is natural to ask whether this source is unique. Does the finite projective behavior just described imply the corresponding local determinacy hypothesis, or can the same behavior be obtained over a weaker mouse?

The global antecedent of this recognition question is Woodin's twelfth Delfino problem. In one formulation, the problem asks whether ZFC together with the statements that every projective set is Lebesgue measurable and has the Baire property, and that every projective relation admits a projective uniformization, implies PD. Woodin showed that this theory implies that $x^\dagger$ exists for every real x and conjectured a positive answer. Steel gave a negative solution in 1997, and Schindler later determined the precise consistency strength of the theory; see [10, 11, 1]. The present paper studies the corresponding finite recognition problem.

Friedman and Schindler isolated finite-level versions of this question in [2]. Motivated by their formulation, we consider the following exact-placement local form.

Question 1 (Friedman–Schindler local problem, exact-placement form)**.** *Let $2 \leqslant n < \omega$. Suppose that every boldface $\boldsymbol{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable and has the Baire property, and suppose that the reals admit a Δ^1_{n+3}-definable well-order. Must $\boldsymbol{\Delta}^1_{n+1}$-determinacy hold?*

Friedman and Schindler obtained a negative answer to a nearby local problem. Working over inner models with finitely many strong cardinals, they produced forcing extensions in which the relevant finite level of the projective hierarchy is universally Baire and the reals admit a projective well-order. Their construction gives the well-order at the desired level from a specific coding real as a parameter, and parameter-free at a higher projective level. Thus the exact lightface placement in Question 1, with a Δ^1_{n+3} well-order and no additional real parameter, remained open.

We resolve this exact-placement problem and add an adjacent uniformization conclusion. Throughout the introduction, uniformization statements written without the word "boldface" are meant in the lightface sense: a pointclass Γ has uniformization if every lightface relation in Γ has a uniformizing function whose graph is again in Γ. Boldface regularity assertions are stated explicitly.

Theorem 1.1. *Let $1 \leqslant n < \omega$, and assume that M_n, the canonical inner model with n Woodin cardinals, exists. Then there is a forcing extension W^* of M_n preserving all cardinals such that $2^{\aleph_0} = \aleph_2$ and:*

1. *every boldface $\boldsymbol{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable;*
2. *every boldface $\boldsymbol{\Sigma}^1_{n+2}$ set of reals has the Baire property;*
3. *the Σ^1_{n+2}-uniformization property holds;*
4. *the Π^1_{n+3}-uniformization property holds;*
5. *the reals admit a Δ^1_{n+3}-definable well-order.*

For $n \geqslant 2$, Theorem 1.1 gives a negative answer to Question 1. The proof records that the final extension does not satisfy $\boldsymbol{\Delta}^1_{n+1}$-determinacy. Hence the regularity-and-well-ordering window suggested by M_{n+1} can already be realized by forcing over M_n. Moreover, the construction gives the two additional lightface uniformization conclusions Σ^1_{n+2}-uniformization and Π^1_{n+3}-uniformization.

The first case, $n = 1$, is not an instance of Question 1 as stated, but it illustrates the same phenomenon at the lowest level covered by the theorem. Starting from M_1, we obtain a model in which all boldface $\boldsymbol{\Sigma}^1_3$ sets are Lebesgue measurable and have the Baire property, Σ^1_3-uniformization and Π^1_4-uniformization hold, and the reals have a Δ^1_4-definable well-order. Thus adjacent Π-uniformization can coexist with a projective well-order of the reals.

Relation with the companion paper. The present paper is the Π-side construction for the local Delfino problem. Its additional conclusion, beyond the exact-placement negative answer, is the adjacent Π^1_{n+3}-uniformization theorem. The forcing mechanism introduced for this purpose is the derivative hierarchy of allowable hybrid forcings, which controls the stability of proposed values for sections of Π^1_{n+3} relations through all later allowable extensions.

The companion paper [6] treats a different direction. It obtains the same $\boldsymbol{\Sigma}^1_{n+2}$ regularity and a Δ^1_{n+3} well-order, but combines them with Martin's Axiom and with a global tail of Σ-uniformization, namely Σ^1_{n+2+m}-uniformization for every $m \in \omega$. That construction is based on coding over codes. The two papers share the same exact-placement background, but their forcing mechanisms are separate: the present construction is based on the local coding predicate and the derivative hierarchy which yields the Π^1_{n+3}-uniformization conclusion.

Proof overview. The construction has four main components. First, over M_n we add branches through an M_n-definable independent sequence of Suslin trees and define a local coding predicate Φ_n of complexity Σ^1_{n+3}. The predicate is arranged to be exact on the coding tags used for the well-order and for the uniformization requirements.

Second, we use a hybrid forcing which combines a countable-support product of M_n-Cohen reservoir coordinates with a finite-support c.c.c. iteration. The reservoir coordinates provide fresh coding areas, while the iteration meets the coding, uniformization, and regularity requirements. This hybrid form refines the forcing methods from [5].

Third, the adjacent Π^1_{n+3}-uniformization conclusion is obtained from a derivative hierarchy of allowable hybrid forcings, refining the method of [4]. At each derivative step, the construction tests whether a proposed value for a section of a Π^1_{n+3} relation is stable under all further allowable extensions or can still be destroyed. The process is iterated to a stable class of allowable forcings, and the final iteration uses only forcings from this stable class.

The same coding predicate gives the Δ^1_{n+3} well-order. For each pair of reals, the construction codes the relative order of their canonical localized presentations. Exactness of Φ_n on the well-order tags ensures that exactly one of the two possible order tags is coded. This gives both a Σ^1_{n+3} and a Π^1_{n+3} definition of the order.

Finally, the regularity bookkeeping is kept separate from the coding bookkeeping. The final iteration places ordinary Cohen stages cofinally often and, for every real parameter a, places relative random, measure-amoeba, and category-amoeba stages over $L[T_{n+1}, a]$ cofinally often. The covering statements obtained from the amoeba stages, together with the small-generic absoluteness of the weakly homogeneous Martin–Solovay tree T_{n+1}, yield Lebesgue measurability and the Baire property for all boldface $\boldsymbol{\Sigma}^1_{n+2}$ sets by Hjorth's argument. The Σ^1_{n+2}-uniformization conclusion is obtained separately from Steel's relativized capture analysis of $M_n(s)$ in the small generic extensions used in the construction.

Organization. Section 2 recalls the inner-model-theoretic background: the relevant fragments of Steel's comparison theory for M_1 and M_n, the recovery of the initial segments used in the construction, and the Martin–Solovay trees T_{n+1}. The next sections define the local coding predicate, the hybrid forcings, and the derivative hierarchy of allowable forcings. The main construction is first carried out in full detail over M_1. The final section records the uniform lift to M_n and proves Theorem 1.1 in full generality.

2 Canonical inner models with Woodin cardinals and the trees T_n

2.1 The canonical inner model with one Woodin cardinal

We recall the fragment of Steel's comparison theory which will be used later. The ground model for the main construction is the canonical proper class mouse M_1, the minimal iterable proper class premouse with one Woodin cardinal. We use the notation and comparison conventions of Steel's outline of inner model theory, and the projective definability analysis of Steel's work on projectively well-ordered inner models; see [12, 13]. In particular, every proper initial segment of M_1 is 1-small and ω-sound, and the order of construction gives a canonical Δ^1_3 well-order of the reals of M_1.

A premouse $\mathcal{M}$ is 1-*small above* η if whenever E is an extender on the $\mathcal{M}$-sequence and $\eta < \mathrm{crit}(E)$, the initial segment $\mathcal{J}^{\mathcal{M}}_{\mathrm{crit}(E)}$ has no Woodin cardinal above η. We say that $\mathcal{M}$ is 1-*small* if it is 1-small above 0.

We also recall the weak iterability notion used by Steel at the first odd level. Let $\mathcal{T}$ be an ω-maximal putative iteration tree on $\mathcal{M}$, let b be a maximal branch through $\mathcal{T}$, and let α be a countable ordinal. The branch b is α-*good* if whenever $\mathcal{N}$ is either $\mathcal{M}^{\mathcal{T}}_b$ itself, or the α-th linear iterate of an initial segment $\mathcal{P} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ by one extender on the $\mathcal{P}$-sequence and its images, then either $\mathcal{N}$ is well-founded or $\alpha \in \mathrm{wfp}(\mathcal{N})$. A countable premouse $\mathcal{M}$ is Π^1_2-*iterable* if player II wins the corresponding one-round weak iteration game: player I plays a countable putative ω-maximal tree together with a countable ordinal α, and player II either accepts a last well-founded model or plays a maximal α-good branch. Steel proves that this iterability condition is Π^1_2 in the codes. For 1-small mice it is the $n = 1$ instance of the general Π_n-iterability analysis; see [13, Lemma 1.7].

We now fix the local notation for limit length trees.

Definition 2.1 (The common part of a limit tree)**.** *Let $\mathcal{T}$ be a k-maximal iteration tree of limit length on a premouse $\mathcal{M}$, where $k \leqslant \omega$.*

1. *We set*
$$\delta(\mathcal{T}) = \sup\{\mathrm{lh}(E^{\mathcal{T}}_\xi) \mid \xi + 1 < \mathrm{lh}(\mathcal{T})\}.$$
2. *$\mathcal{M}(\mathcal{T})$ denotes the common part of the models along $\mathcal{T}$ below $\delta(\mathcal{T})$, i.e. the unique passive premouse $\mathcal{P}$ of height $\delta(\mathcal{T})$ such that, for every extender $E^{\mathcal{T}}_\xi$ used in $\mathcal{T}$, $\mathcal{P}$ agrees with the corresponding model of the tree below* $\mathrm{lh}(E^{\mathcal{T}}_\xi)$*.*

We shall use the following form of Steel's branch uniqueness theorem, often called the zipper lemma.

Theorem 2.2 (Steel's zipper lemma)**.** *Let $\mathcal{T}$ be a k-maximal iteration tree of limit length on a premouse $\mathcal{M}$, where $k \leqslant \omega$, and let b, c be distinct* *cofinal branches through $\mathcal{T}$. Put $\delta = \delta(\mathcal{T})$. Suppose that $A \subseteq \delta$ and that $A, \delta \in \mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_b) \cap \mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_c)$. Then one of the two branch models satisfies*

$$\exists\kappa < \delta\,(\kappa \textit{ is } A\textit{-strong up to } \delta).$$

Definition 2.3 ($\mathcal{Q}$-structures)**.** *Let $\mathcal{T}$ be a k-maximal iteration tree of limit length on a premouse $\mathcal{M}$, where $k \leqslant \omega$, and let b be a cofinal well-founded branch through $\mathcal{T}$. The $\mathcal{Q}$-structure $\mathcal{Q}(b, \mathcal{T})$ is the least initial segment of $\mathcal{M}^{\mathcal{T}}_b$, if it exists, which either sees that $\delta(\mathcal{T})$ is not Woodin over the common part $\mathcal{M}(\mathcal{T})$, or projects strictly below $\delta(\mathcal{T})$ at an allowed finite degree. More explicitly, $\mathcal{Q}(b, \mathcal{T}) = \mathcal{J}^{\mathcal{M}^{\mathcal{T}}_b}_\gamma$ for the least γ such that either*

$$\mathcal{J}^{\mathcal{M}^{\mathcal{T}}_b}_{\gamma+1} \models \text{“}\delta(\mathcal{T}) \textit{ is not Woodin}\text{''},$$

or, for some $m < \omega$ allowed by the degree of the tree,

$$\rho_{m+1}(\mathcal{J}^{\mathcal{M}^{\mathcal{T}}_b}_\gamma) < \delta(\mathcal{T}).$$

If no such γ exists, then $\mathcal{Q}(b, \mathcal{T})$ is undefined.

The point of the preceding definition is that, in the 1-small context, a $\mathcal{Q}$-structure determines at most one good cofinal branch. If two distinct branches had the same relevant well-founded $\mathcal{Q}$-structure, the zipper lemma would produce strength below $\delta(\mathcal{T})$, contradicting the initial segment which witnesses that $\delta(\mathcal{T})$ is not Woodin.

In the next few statements, an *ordinary premouse* means a premouse in Steel's usual premouse language, with no real parameter, predicate parameter, or base set added to the structure. This is only a terminological convention distinguishing these premice from mice over a real or over another base; no additional fine-structural notion is being introduced.

Lemma 2.4 (Comparison with a Π^1_2-iterable mouse)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be countable ordinary premice. Assume that both are ω-sound and project to ω, that $\mathcal{M} \lhd M_1$, and that $\mathcal{N}$ is 1-small and Π^1_2-iterable. Then the comparison of $\mathcal{M}$ with $\mathcal{N}$ is successful. Consequently*

$$\mathcal{M} \trianglelefteq \mathcal{N} \quad \textit{or} \quad \mathcal{N} \trianglelefteq \mathcal{M}.$$

Proof. Run the usual coiteration by least disagreement, producing trees $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$. The $\mathcal{M}$-side is governed by the strategy inherited from M_1. The $\mathcal{N}$-side is governed by the winning strategy witnessing Π^1_2-iterability.

There is no new issue at successor stages. Consider a countable limit stage and suppose first that both sides have reached the same comparison height, so that $\delta(\mathcal{T}) = \delta(\mathcal{U})$. Let b be the branch selected on the $\mathcal{M}$-side. Since $\mathcal{M} \lhd M_1$ and M_1 is 1-small below its Woodin, the branch model $\mathcal{M}^{\mathcal{T}}_b$ has a $\mathcal{Q}$-structure witnessing the relevant failure of Woodinness at this common

δ. Write this $\mathcal{Q}$-structure as $L_\alpha(\mathcal{M}(\mathcal{T}))$, using the standard coding of the common part.

Apply the Π^1_2-iterability of $\mathcal{N}$ to the tree $\mathcal{U}$ with ordinal parameter α. Player II supplies a maximal α-good branch c. Because the two sides of the comparison agree below the common δ, the initial segment $L_\alpha(\mathcal{M}(\mathcal{T}))$ is also the corresponding $\mathcal{Q}$-structure for c on the $\mathcal{N}$-side, provided the branch model is well-founded past that structure. If there were a second sufficiently good cofinal branch through $\mathcal{U}$, then the zipper lemma, applied with a code for the common $\mathcal{Q}$-structure as parameter A, would produce strength below $\delta(\mathcal{U})$. This contradicts the witness to the failure of Woodinness coded by the $\mathcal{Q}$-structure. Hence the good branch through $\mathcal{U}$ is unique, and it is fully well-founded.

If one side has stopped using extenders, the same argument applies with the last model on the stopped side replacing the common branch model. Its relevant initial segment supplies the $\mathcal{Q}$-structure which witnesses that the remaining comparison height is not Woodin, and therefore determines the unique well-founded branch on the other side.

It remains to rule out a comparison of length ω_1. Suppose that such a putative comparison existed. Force with the Lévy collapse $\mathrm{Col}(\omega, \omega_1)$ over the ambient model. In the collapse extension, the putative tree becomes countable, and Π^1_2-iterability is preserved. Thus there is a branch supplied by the Π^1_2 strategy. By the uniqueness just proved, this branch is ordinal definable from the ground-model data of the comparison. Homogeneity of the collapse therefore puts the branch back in the ground model, contradicting the assumption that the comparison had no branch at stage ω_1.

Thus the comparison terminates below ω_1. The terminal models are linearly ordered by initial segment. Since both premice are ω-sound and project to ω, the usual no-drop argument for the shorter side pulls the conclusion back to the original premice. Hence $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$. $\square$

Preservation convention for the outer models used below. We shall use Lemma 2.4 only in the forcing extensions which occur in this paper. The preservation fact needed there is the following: if $\mathcal{M} \triangleleft M_1$ is countable and belongs to one of these extensions, then $\mathcal{M}$ remains Π^1_2-iterable there. This is the realizability preservation supplied by Steel's weak iteration-game analysis for initial segments of M_1, applied to the proper, ω_1-preserving forcing extensions used below. We do not claim that the set of all real codes for Π^1_2-iterable premice is absolute between arbitrary outer models with the same ω_1.

Lemma 2.5 (Definable cofinal system of M_1-initial segments)**.** *Let $M_1[G]$ be an ω_1-preserving forcing extension of M_1 of the kind fixed in Paragraph 2.1.*

In $M_1[G]$ *let*

$$\begin{aligned}\mathcal{I} = \big\{\mathcal{N} \mid \mathcal{N} &\textit{ is a countable ordinary premouse,}\\ \mathcal{N} &\textit{ is } 1\textit{-small, } \mathcal{N} \textit{ is } \omega\textit{-sound,}\\ \rho_\omega(\mathcal{N}) &= \omega, \textit{ and } \mathcal{N} \textit{ is } \Pi^1_2\textit{-iterable}\big\}.\end{aligned}$$

Then $\mathcal{I}$ *is* Π^1_2*-definable in the codes. Moreover every member of* $\mathcal{I}$ *is of the form* $\mathcal{J}^{M_1}_\eta$ *for some* $\eta < \omega_1$*, and*

$$\{\eta < \omega_1 \mid \mathcal{J}^{M_1}_\eta \in \mathcal{I}\}$$

is cofinal in ω_1*.*

Proof. The clauses saying that a real codes a countable ordinary premouse, that the premouse is 1-small, that it is ω-sound, and that $\rho_\omega = \omega$ are arithmetic, after fixing the usual coding of countable premice by reals. By Steel's analysis of the weak iteration games, Π^1_2-iterability is a Π^1_2 condition in the code. Hence the displayed definition of $\mathcal{I}$ is Π^1_2.

We next show that no nonstandard premouse enters $\mathcal{I}$. Work in $M_1[G]$ and let $\mathcal{N} \in \mathcal{I}$. Since ω_1 is preserved, $\mathcal{N}$ has countable height below the true $\omega_1^{M_1}$. Choose $\eta < \omega_1$ such that $\mathrm{Ord}^{\mathcal{N}} < \eta$ and such that $\mathcal{J}^{M_1}_\eta$ is ω-sound and projects to ω. The initial segment $\mathcal{J}^{M_1}_\eta$ is an ordinary premouse and is realizable inside M_1; hence it is Π^1_2-iterable in the present extension by Paragraph 2.1. Applying Lemma 2.4 to $\mathcal{J}^{M_1}_\eta$ and $\mathcal{N}$, the alternative $\mathcal{J}^{M_1}_\eta \lhd \mathcal{N}$ is impossible by the choice of η. Therefore $\mathcal{N} \trianglelefteq \mathcal{J}^{M_1}_\eta$, so $\mathcal{N}$ is itself an initial segment of M_1.

Finally, the fine structure of M_1 gives cofinally many $\eta < \omega_1$ such that $\mathcal{J}^{M_1}_\eta$ is ω-sound and projects to ω. These levels are ordinary premice, 1-small, and realizable, hence Π^1_2-iterable in the present extension by the same preservation convention. Therefore they belong to $\mathcal{I}$, proving cofinality. □

Definition 2.6 (Recovering $M_1|\omega_1$)**.** *In any* ω_1*-preserving forcing extension of* M_1 *of the kind fixed in Paragraph 2.1, we write*

$$\mathcal{N} = M_1|\omega_1$$

for the assertion that

$$\mathcal{N} = \bigcup\{\mathcal{M} \mid \mathcal{M} \in \mathcal{I}\},$$

where $\mathcal{I}$ *is the class from Lemma 2.5. Equivalently,* $x \in \mathcal{N}$ *iff* x *belongs to some countable ordinary premouse which is* 1*-small,* ω*-sound,* Π^1_2*-iterable, and projects to* ω*.*

Lemma 2.7 (The recovered initial segment)**.** *Let* $M_1[G]$ *be an* ω_1*-preserving forcing extension of* M_1 *of the kind fixed in Paragraph 2.1. Then Definition 2.6 defines the true initial segment*

$$M_1|\omega_1 = \mathcal{J}^{M_1}_{\omega_1}.$$

Moreover the definition is uniform in all such extensions.

Proof. By Lemma 2.5, every member of $\mathcal{I}$ is an initial segment $\mathcal{J}_\eta^{M_1}$ with $\eta < \omega_1$. Hence the union in Definition 2.6 is contained in $M_1|\omega_1$. Conversely, the same lemma gives cofinally many $\eta < \omega_1$ with $\mathcal{J}_\eta^{M_1} \in \mathcal{I}$. Their union is $\mathcal{J}_{\omega_1}^{M_1}$. The same formula defining $\mathcal{I}$ is used in every such extension. We do not assert that the same real codes belong to $\mathcal{I}$ in different extensions, since new reals may code new countable putative iteration trees. Rather, applying Lemma 2.5 inside the given extension identifies the union of the premice satisfying that formula with the true $\mathcal{J}_{\omega_1}^{M_1}$. □

Remark. The notation in Definition 2.6 is a convention for outer models of M_1 which preserve ω_1. The same first-order-looking assertion, if evaluated inside an arbitrary transitive model, need not imply that the object obtained is the true $M_1|\omega_1$. Later, whenever this definition is used inside countable auxiliary models, the relevant countable premouse is also required externally to belong to the class $\mathcal{I}$.

We shall need a canonical diamond sequence which is available from the same fine structure. The argument is Jensen's proof of diamond in L, with Steel's condensation theorem for initial segments of M_1 replacing condensation for L.

Theorem 2.8 (Steel condensation, in the form used here)**.** *Let $\mathcal{M} \trianglelefteq M_1$ be an ω-sound initial segment and let*

$$\pi : \bar{N} \to \mathcal{M}$$

be the inverse of the transitive collapse of a sufficiently elementary substructure of $\mathcal{M}$. Suppose that the critical point of π is the relevant standard projectum of $\bar{N}$. Then either

1. *$\bar{N} \trianglelefteq \mathcal{M}$, or*
2. *$\bar{N}$ is an initial segment of a degree-zero ultrapower of an initial segment of $\mathcal{M}$ by an extender on the M_1-sequence whose length is that projectum.*

In the hulls used in the diamond argument below, the second alternative is impossible. Hence the transitive collapse is an initial segment of M_1.

Proof. This is the standard condensation theorem for the Steel M_1-construction; see [12, Theorem 5.1]. We only spell out why the ultrapower alternative does not occur in the present application. The hulls below are chosen so that their collapse has projectum equal to its internal ω_1. If the second alternative held, there would be an extender on the M_1-sequence indexed exactly at this internal ω_1. The lower part below that index is the collapse of the corresponding lower part of the hull and sees the index as the successor cut reached by the construction. An extender indexed there would make the index inaccessible in the relevant extender model. This contradicts the fact that the collapsed lower part computes it as its ω_1. Therefore only the initial-segment alternative remains. $\square$

Lemma 2.9 (A canonical M_1-diamond sequence)**.** *There is a sequence*

$$\vec{D} = \langle D_\alpha \mid \alpha < \omega_1 \rangle \in M_1$$

with $D_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$ such that

$$M_1 \models \text{“}\vec{D} \textit{ is a } \Diamond_{\omega_1}\textit{-sequence''}.$$

Moreover $\vec{D}$ is uniformly definable over $M_1|\omega_1$ from the canonical order of construction of M_1.

Proof. Work in M_1. Use the canonical well-order $<_{M_1}$ of the construction to define $\vec{D}$ recursively. At a limit ordinal $\alpha < \omega_1$, suppose $\langle D_\beta \mid \beta < \alpha \rangle$ has been defined. If there is a pair (A, C) such that $A \subseteq \alpha$, $C \subseteq \alpha$ is club in α, and

$$\forall \beta \in C\, (D_\beta \neq A \cap \beta),$$

then let (A_α, C_α) be the $<_{M_1}$-least such pair and set $D_\alpha = A_\alpha$. If there is no such pair, set $D_\alpha = \varnothing$. At successor stages we may again set $D_\alpha = \varnothing$.

The recursion is carried out over $M_1|\omega_1$. Indeed, all objects considered at stage α are subsets of the countable ordinal α in M_1, and the order used to choose the least pair is the restriction of the canonical M_1 construction order. Hence the resulting sequence is uniformly definable over $M_1|\omega_1$.

It remains to verify that $\vec{D}$ is a diamond sequence. Suppose not. Let (A, C) be the $<_{M_1}$-least counterexample, so $A \subseteq \omega_1$, $C \subseteq \omega_1$ is club, and

$$\forall \alpha \in C\, (D_\alpha \neq A \cap \alpha).$$

Choose an ω-sound initial segment $\mathcal{J}_\theta^{M_1}$ containing A, C, and the sequence $\vec{D}$, and then take a countable elementary substructure

$$X \prec \mathcal{J}_\theta^{M_1}$$

with $A, C, \vec{D} \in X$ and with $\alpha = X \cap \omega_1 \in C$. Let

$$\pi : \bar{X} \to X$$

be the inverse of the transitive collapse. By Theorem 2.8, $\bar{X}$ is an initial segment of M_1. Consequently the recursive construction of $\vec{D}$ inside $\bar{X}$ is exactly the initial part

$$\langle D_\beta \mid \beta < \alpha \rangle.$$

Moreover the collapse sends A to $A \cap \alpha$ and C to $C \cap \alpha$, and by elementarity $\bar{X}$ sees $(A \cap \alpha, C \cap \alpha)$ as the $<_{M_1}$-least witness that the previous sequence is not diamond on α. Therefore the recursion at stage α gives

$$D_\alpha = A \cap \alpha.$$

Since $\alpha \in C$, this contradicts the choice of (A, C). Thus no counterexample exists, and $\vec{D}$ is a $\diamondsuit_{\omega_1}$-sequence in M_1. $\square$

We fix once and for all the $<_{M_1}$-least sequence $\vec{D}$ satisfying Lemma 2.9. In later sections this sequence will be used to define a canonical ω_1-sequence of independent Suslin trees over M_1. The construction of those trees is postponed until the point where the coding machinery needs them.

2.2 The canonical inner model with n-many Woodin cardinals

We now record the higher-level analogue of Subsection 2.1. Throughout this subsection $1 \leqslant n < \omega$ is fixed, and M_n denotes the canonical minimal proper class premouse with n Woodin cardinals, in the sense of Steel's construction of tame mice with full background extenders. Thus $M_0 = L$, and for $n > 0$ the model M_n is obtained from the Steel background construction by stopping at the first failure of n-smallness, or as the limit of the n-small construction if no such failure occurs. We shall use only the following standard consequences of Steel's analysis.

First, if there are n Woodin cardinals, then M_n exists and satisfies that there are n Woodin cardinals. Secondly, every proper initial segment of M_n is n-small and ω-sound. Thirdly, the order of construction of M_n gives a canonical construction well-order of the reals of M_n, and Steel's projective analysis shows that this well-order is Δ^1_{n+2} over the reals of M_n; see [13, 12]. The case $n = 1$ is exactly the situation isolated in the previous subsection.

Definition 2.10 (n-smallness)**.** *Let $\mathcal{M}$ be a premouse and let $\eta < \mathrm{Ord}^{\mathcal{M}}$. We say that $\mathcal{M}$ is* n-small above η *if whenever E is an extender on the $\mathcal{M}$-sequence and*

$$\eta < \mathrm{crit}(E),$$

then the initial segment $\mathcal{J}^{\mathcal{M}}_{\mathrm{crit}(E)}$ does not have n Woodin cardinals above η. We say that $\mathcal{M}$ is n-small *if it is n-small above* 0.

We shall also use Steel's finite-level iterability condition. The exact definition depends on the parity of n. For even n, one uses the n-round weak iteration game in which the branches played by player II are required to be sufficiently definable over the trees played by player I. For odd n, one adds the corresponding α-goodness requirement at the last round. In both cases the definition is arranged so that realizable mice satisfy it, and so that it is projectively simple in the codes.

Definition 2.11 (Steel Π_n-iterability)**.** *Let $\mathcal{M}$ be a countable premouse and let $\eta < \mathrm{Ord}^{\mathcal{M}}$ be a cutpoint. We say that $\mathcal{M}$ is* Π_n-iterable above η *if player II has a winning strategy in Steel's game $\mathcal{G}(\mathcal{M}, \eta, n)$ for Π_n-iterability above η. We say simply that $\mathcal{M}$ is* Π_n-iterable *if it is Π_n-iterable above 0.*

For later reference we isolate the three consequences of the definition which are used in this paper.

Fact 2.12 (Steel)**.** *For each fixed $1 \leqslant n < \omega$ the following hold.*

1. *The relation*

   $$\text{“}x \textit{ codes a countable premouse which is } \Pi_n\textit{-iterable}''$$

   is $\mathbf{\Pi}^1_{n+1}$ in the real code x.

2. *If $\mathcal{M}$ is a countable initial segment of M_n, then $\mathcal{M}$ is Π_n-iterable in the Steel sense, in all outer models considered in this paper in which the relevant realizability argument is preserved.*

3. *If $\mathcal{M}$ is countable, n-small and realizable, and $\mathcal{N}$ is countable, n-small and Π_n-iterable, then the Steel comparison of $\mathcal{M}$ and $\mathcal{N}$ is successful, provided the two mice have the same lower part. In the applications below the lower part is empty, so this is the $\eta = 0$ case.*

The next lemma is the direct analogue of Lemma 2.4. We state it separately because it is the form in which it will be used later.

Lemma 2.13 (Comparison with a Π_n-iterable mouse)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be countable ordinary premice. Assume that both are ω-sound and project to ω, that $\mathcal{M} \lhd M_n$, and that $\mathcal{N}$ is n-small and Π_n-iterable. Then the comparison of $\mathcal{M}$ with $\mathcal{N}$ is successful. Consequently*

$$\mathcal{M} \trianglelefteq \mathcal{N} \quad \textit{or} \quad \mathcal{N} \trianglelefteq \mathcal{M}.$$

Proof. This is the $\eta = 0$ instance of Steel's comparison theorem for n-small Π_n-iterable mice. Since $\mathcal{M} \lhd M_n$, the premouse $\mathcal{M}$ is realizable in the background construction of M_n. Since $\mathcal{N}$ is n-small and Π_n-iterable, the $\mathcal{N}$-side has exactly the amount of iterability required by Steel's comparison lemma. Because both premice are ordinary premice in the parameter-free sense fixed above, the lower parts agree trivially.

The proof is the usual least-disagreement coiteration. The $\mathcal{M}$-side uses the realization strategy inherited from the construction of M_n, while the $\mathcal{N}$-side uses the Π_n-iterability strategy. At limit stages, the good branch is characterized by the corresponding $\mathcal{Q}$-structure: for $n = 1$ this is the branch uniqueness argument described in Subsection 2.1, and for $n > 1$ the assertion is proved by induction on n, because the $\mathcal{Q}$-structure appearing at a limit stage is $(n-1)$-small and Π_{n-1}-iterable above the new cutpoint. If two distinct good branches were available, Steel's zipper argument would produce the forbidden strength below the comparison height, contradicting n-smallness at the relevant level. Thus the comparison has well-founded branches and reaches terminal models which are linearly ordered by initial segment.

Finally, since both premice are ω-sound and project to ω, the standard no-drop pullback argument gives the initial-segment relation already between the original premice. Hence $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$. □

Preservation convention for the outer models used below. As in the M_1 case, we shall only use the preceding comparison in forcing extensions which occur in this paper. The preservation fact needed is the following one: if $\mathcal{M} \triangleleft M_n$ is countable and belongs to such an extension, then $\mathcal{M}$ remains Π_n-iterable there. This is the higher-level version of the realizability preservation used in Paragraph 2.1. We do not claim that the class of all real codes for Π_n-iterable premice is absolute between arbitrary outer models with the same ω_1.

For the coding arguments we only need to recover the lower part $M_n|\omega_1$. Thus we isolate the lower-part initial segments of M_n, rather than the larger class of all n-small Π_n-iterable mice. This distinction is harmless for $n=1$, but it is important at higher odd levels, where one has to avoid the familiar nonstandard Π_n-iterable mice. The lower-part restriction below excludes these objects from the definition used in the coding apparatus.

Definition 2.14 (Lower-part M_n-approximations). *Let $M_n[G]$ be an ω_1-preserving forcing extension of M_n of the kind fixed in Paragraph 2.2. In $M_n[G]$ let $\mathcal{I}_n$ be the class of all $\mathcal{N}$ such that*

1. *$\mathcal{N}$ is a countable passive ordinary premouse;*
2. *$\mathcal{N}$ is a lower-part premouse, i.e. $\mathcal{N}$ has no Woodin cardinal;*
3. *$\mathcal{N}$ is n-small, ω-sound, and $\rho_\omega(\mathcal{N}) = \omega$;*
4. *$\mathcal{N}$ is Π_n-iterable.*

Lemma 2.15 (Definable cofinal system of M_n-initial segments). *Let $M_n[G]$ be an ω_1-preserving forcing extension of M_n of the kind fixed above. Then $\mathcal{I}_n$ is $\mathbf{\Pi}^1_{n+1}$-definable in the codes. Moreover every member of $\mathcal{I}_n$ is of the form $\mathcal{J}_\eta^{M_n}$ for some $\eta < \omega_1$, and*

$$\{\eta < \omega_1 \mid \mathcal{J}_\eta^{M_n} \in \mathcal{I}_n\}$$

is cofinal in ω_1.

Proof. The clauses saying that a real codes a countable passive ordinary premouse, that the premouse is lower-part, n-small, ω-sound, and satisfies $\rho_\omega = \omega$, are arithmetic in the usual code for countable premice. By Fact 2.12, the Π_n-iterability clause is $\mathbf{\Pi}^1_{n+1}$. Therefore $\mathcal{I}_n$ is $\mathbf{\Pi}^1_{n+1}$ in the codes.

We next prove that no nonstandard lower-part premouse enters $\mathcal{I}_n$. Work in $M_n[G]$ and let $\mathcal{N} \in \mathcal{I}_n$. Since ω_1 is preserved, the height of $\mathcal{N}$ is below the true $\omega_1^{M_n}$. Choose $\eta < \omega_1$ such that $\mathrm{Ord}^{\mathcal{N}} < \eta$ and such that $\mathcal{J}_\eta^{M_n}$ is passive, lower-part, ω-sound, and projects to ω. The fine structure of M_n gives cofinally many such η. The initial segment $\mathcal{J}_\eta^{M_n}$ is realizable inside M_n, and hence is Π_n-iterable in the present extension by the preservation convention.

Apply Lemma 2.13 to $\mathcal{J}_\eta^{M_n}$ and $\mathcal{N}$. The alternative $\mathcal{J}_\eta^{M_n} \trianglelefteq \mathcal{N}$ is impossible by the choice of η, because $\mathrm{Ord}^{\mathcal{N}} < \eta$. Therefore

$$\mathcal{N} \trianglelefteq \mathcal{J}_\eta^{M_n},$$

and hence $\mathcal{N}$ is itself an initial segment of M_n.

Conversely, let $\eta < \omega_1$ be such that $\mathcal{J}_\eta^{M_n}$ is passive, lower-part, ω-sound, and projects to ω. Then $\mathcal{J}_\eta^{M_n}$ is n-small and realizable in the Steel construction of M_n, and by the preservation convention it is Π_n-iterable in $M_n[G]$. Thus $\mathcal{J}_\eta^{M_n} \in \mathcal{I}_n$. Since such η are cofinal in ω_1, the cofinality assertion follows. ☐

Definition 2.16 (Recovering $M_n|\omega_1$)**.** *In any ω_1-preserving forcing extension of M_n of the kind fixed above, we write*

$$\mathcal{N} = M_n|\omega_1$$

if

$$\mathcal{N} = \bigcup \mathcal{I}_n,$$

where $\mathcal{I}_n$ is the class from Definition 2.14, computed in that extension.

Lemma 2.17 (Correctness of the recovery)**.** *Let $M_n[G]$ be an ω_1-preserving forcing extension of M_n of the kind fixed above. Then Definition 2.16 defines the true initial segment*

$$M_n|\omega_1 = \mathcal{J}_{\omega_1}^{M_n}.$$

Moreover the definition is uniform in all such extensions.

Proof. By Lemma 2.15, every member of $\mathcal{I}_n$ is an initial segment $\mathcal{J}_\eta^{M_n}$ with $\eta < \omega_1$. Hence the union in Definition 2.16 is contained in $M_n|\omega_1$. Conversely, the same lemma gives cofinally many $\eta < \omega_1$ with $\mathcal{J}_\eta^{M_n} \in \mathcal{I}_n$. The union of these initial segments is $\mathcal{J}_{\omega_1}^{M_n}$.

The same formula defining $\mathcal{I}_n$ is used in every relevant extension. We do not assert that the same real codes belong to $\mathcal{I}_n$ in different extensions, since new reals may code new countable putative iteration trees. Rather, applying Lemma 2.15 inside the given extension identifies the union of the premice satisfying the formula with the true $\mathcal{J}_{\omega_1}^{M_n}$. ☐

Remark. The notation $M_n|\omega_1$ in Definition 2.16 is a convention for the outer models of M_n used in this paper. It should not be read as saying that an arbitrary transitive model which internally satisfies the same first-order-looking definition has computed the true $M_n|\omega_1$. Later, whenever countable auxiliary models are used, the relevant countable premouse is also required externally to belong to $\mathcal{I}_n$.

The condensation and diamond arguments from the previous subsection lift without change once M_1 is replaced by M_n.

Theorem 2.18 (Steel condensation for M_n, in the form used here)**.** *Let $\mathcal{M} \trianglelefteq M_n$ be an ω-sound initial segment and let*

$$\pi : \bar{N} \to \mathcal{M}$$

be the inverse of the transitive collapse of a sufficiently elementary substructure of $\mathcal{M}$. Suppose that the critical point of π is the relevant standard projectum of $\bar{N}$. Then either

1. *$\bar{N} \trianglelefteq \mathcal{M}$, or*
2. *$\bar{N}$ is an initial segment of a degree-zero ultrapower of an initial segment of $\mathcal{M}$ by an extender on the M_n-sequence whose length is that projectum.*

In the hulls used in the diamond argument below, the second alternative is impossible. Hence the transitive collapse is an initial segment of M_n.

Proof. This is the condensation theorem for initial segments of the Steel M_n-construction. The proof is the same fine-structural argument as in the M_1 case, with n-smallness replacing 1-smallness. The only point needed below is the exclusion of the ultrapower alternative. The hulls in the diamond argument are chosen so that the collapsed structure has projectum equal to its internal ω_1. If the ultrapower alternative occurred, an extender on the M_n-sequence would be indexed at this internal ω_1. The lower part below that index computes the index as its first uncountable cardinal, while the presence of such an extender would make it inaccessible in the relevant extender model. This contradiction leaves only the initial-segment alternative. ☐

Fact 2.19 (Steel's projective well-order of M_n)**.** *The construction order $<_{M_n}$ restricted to the reals of M_n is a Δ^1_{n+2} well-order. Equivalently, for reals $x, y \in M_n$, the assertion that x is constructed before y in M_n is given by a Σ^1_{n+2} formula and also by a Π^1_{n+2} formula.*

We shall use this fact only as a source of canonical choices inside M_n. In particular, using $<_{M_n}$ we fix once and for all the canonical almost disjoint family

$$D = \langle d_\xi \mid \xi < \omega_1 \rangle \in M_n$$

used for almost disjoint coding, and the canonical $\diamondsuit_{\omega_1}$- sequence used in the bookkeeping. No later argument uses the well-order to choose elements from projective sections in the final model.

Lemma 2.20 (A canonical M_n-diamond sequence)**.** *There is a sequence*

$$\vec{D}^n = \langle D^n_\alpha \mid \alpha < \omega_1 \rangle \in M_n$$

with $D^n_\alpha \subseteq \alpha$ *for all* $\alpha < \omega_1$ *such that*

$$M_n \models \text{“}\vec{D}^n \textit{ is a } \diamondsuit_{\omega_1}\textit{-sequence}\text{''}.$$

Moreover $\vec{D}^n$ *is uniformly definable over* $M_n|\omega_1$ *from the canonical order of construction of* M_n*.*

Proof. Work in M_n. Use the canonical well-order $<_{M_n}$ of the construction to define $\vec{D}^n$ recursively. At a limit ordinal $\alpha < \omega_1$, suppose $\langle D^n_\beta \mid \beta < \alpha \rangle$ has already been defined. If there is a pair (A, C) such that $A \subseteq \alpha$, $C \subseteq \alpha$ is club in α, and

$$\forall \beta \in C \, (D^n_\beta \neq A \cap \beta),$$

then let (A_α, C_α) be the $<_{M_n}$-least such pair and set $D^n_\alpha = A_\alpha$. If there is no such pair, set $D^n_\alpha = \varnothing$. At successor stages we again set $D^n_\alpha = \varnothing$.

The recursion is carried out over $M_n|\omega_1$, because at stage α all relevant objects are subsets of the countable ordinal α in M_n, and the order used to choose the least pair is the restriction of the canonical construction order of M_n.

Suppose toward a contradiction that $\vec{D}^n$ is not a diamond sequence in M_n. Let (A, C) be the $<_{M_n}$-least counterexample, so $A \subseteq \omega_1$, $C \subseteq \omega_1$ is club, and

$$\forall \alpha \in C \, (D^n_\alpha \neq A \cap \alpha).$$

Choose an ω-sound initial segment $\mathcal{J}^{M_n}_\theta$ containing A, C, and the sequence $\vec{D}^n$, and take a countable elementary substructure

$$X \prec \mathcal{J}^{M_n}_\theta$$

with $A, C, \vec{D}^n \in X$ and with $\alpha = X \cap \omega_1 \in C$. Let

$$\pi : \bar{X} \to X$$

be the inverse of the transitive collapse. By Theorem 2.18, the collapse $\bar{X}$ is an initial segment of M_n. Therefore the recursive construction of $\vec{D}^n$ inside $\bar{X}$ is exactly the initial part

$$\langle D^n_\beta \mid \beta < \alpha \rangle.$$

The collapse sends A to $A \cap \alpha$ and C to $C \cap \alpha$, and by elementarity $\bar{X}$ sees $(A \cap \alpha, C \cap \alpha)$ as the $<_{M_n}$-least witness that the previous sequence is not diamond on α. Hence the recursion at stage α gives

$$D^n_\alpha = A \cap \alpha.$$

This contradicts $\alpha \in C$. Therefore no counterexample exists, and $\vec{D}^{\,n}$ is a $\diamondsuit_{\omega_1}$-sequence in M_n. $\square$

We fix once and for all the $<_{M_n}$-least sequence $\vec{D}^{\,n}$ satisfying Lemma 2.20. In the higher-level version of the construction, this sequence plays exactly the role played by the M_1-diamond sequence in the detailed M_1 case: it gives the canonical lower-part bookkeeping from which the corresponding sequence of independent Suslin trees and the localized coding apparatus are built.

2.3 The trees T_n, weak homogeneity, and small generic absoluteness

We next fix the tree representations of the projective pointclasses which will be used in the regularity argument. The main application in this paper is the case $n = 2$: the Martin–Solovay tree $T_2 \in M_1$ represents the universal $\boldsymbol{\Sigma}^1_3$ set. Since the higher-level notation is no harder, we record the construction uniformly. Thus, for $n \geqslant 2$, the tree T_n will be a canonical tree belonging to M_{n-1} whose projection is the universal $\boldsymbol{\Sigma}^1_{n+1}$ set of reals. In the final model we shall use only the following consequence of its construction: membership in $p[T_n]$ is absolute to the small generic extensions in which the relevant real appears.

We recall the form of weak homogeneity used below. The notation follows Steel's exposition of the Martin–Solovay theorem in terms of towers of measures [14]. If Z is a set and κ is a cardinal, let $\mathrm{meas}_\kappa(Z)$ be the set of κ-additive measures on $Z^{<\omega}$. If $\mu \in \mathrm{meas}_\kappa(Z)$, its dimension is the unique $m < \omega$ such that μ concentrates on Z^m. If μ has dimension m, ν has dimension k, and $m \leqslant k$, we say that ν projects to μ if for every $A \subseteq Z^m$,

$$A \in \mu \quad \Longleftrightarrow \quad \{u \in Z^k \mid u \upharpoonright m \in A\} \in \nu.$$

A tower $\langle \mu_i \mid i < \omega \rangle$ is countably complete if, whenever $A_i \in \mu_i$ for every i, there is an $f \in Z^\omega$ such that $f \upharpoonright \dim(\mu_i) \in A_i$ for every i.

Definition 2.21 (Weakly homogeneous tree)**.** *Let T be a tree on $\omega \times Z$. For $x \in \omega^\omega$ write*

$$T_x = \{u \in Z^{<\omega} \mid \forall k < |u| \; ((x \upharpoonright k, u \upharpoonright k) \in T)\}.$$

We say that T is κ-weakly homogeneous *if there is a countable set*

$$\mathcal{M}_T \subseteq \mathrm{meas}_\kappa(Z)$$

closed under projections such that, for every real x,

$$x \in p[T]$$

if and only if there is a countably complete tower $\langle \mu_i \mid i < \omega \rangle$ from $\mathcal{M}_T$ such that each μ_i concentrates on the corresponding finite section of T_x. Equivalently, T is weakly homogeneous in the sense of a weak homogeneity system whose range is countable.

When defined from such a witnessing system, we call $p[T]$ a κ-weakly homogeneously Suslin set.

Definition 2.22 (Absolute complements and universal Baireness)**.** *Let T be a tree on $\omega \times Z$ and S a tree on $\omega \times Z'$. We say that T and S are κ-*absolute complements *if, for every forcing extension by a partial order of size $< \kappa$,*

$$p[T] = \omega^\omega \backslash p[S].$$

*A set of reals is κ-*universally Baire *if it is the projection of a tree which has a κ-absolute complement.*

The connection between the preceding two notions is the Martin–Solovay tree construction. If $\bar{\mu}$ is a weak homogeneity system for T and Θ is sufficiently large, the Martin–Solovay tree

$$\mathrm{ms}(\bar{\mu}, \Theta)$$

searches for a coherent descending sequence of ordinal ranks through the ultrapowers determined by the measures in $\bar{\mu}$. A branch through this Martin–Solovay tree is precisely a continuous certificate that the corresponding tower is ill-founded. The following is the form of the Martin–Solovay theorem used in the rest of the paper [14, Theorem 2.20 and Corollary 2.21].

Theorem 2.23 (Martin–Solovay, Steel's formulation)**.** *Let T be κ-weakly homogeneous via a weak homogeneity system $\bar{\mu}$, and let $\Theta > |T|^+$. Then T and $\mathrm{ms}(\bar{\mu}, \Theta)$ are κ-absolute complements. In particular, $p[T]$ is κ-universally Baire. Moreover, in every forcing extension by a partial order of size $< \kappa$, the measures in $\bar{\mu}$ lift to a weak homogeneity system witnessing the same assertion for the same ground-model tree T.*

Proof. This is the Martin–Solovay theorem for weakly homogeneous trees. The key point is that a weak homogeneity system gives, for each real x, a countable tree of possible measure towers. If one of these towers is countably complete, then $x \in p[T]$. If no such tower is countably complete, the associated ill-foundedness is witnessed uniformly by a descending sequence of ordinal ranks; these ranks form a branch through $\mathrm{ms}(\bar{\mu}, \Theta)_x$, provided Θ is chosen above the size of the relevant tree.

Small forcing preserves the measure towers in the required way: the measures extend canonically to the forcing extension, and functions in the extension are represented modulo the lifted measures by ground-model functions. Consequently the same Martin–Solovay tree remains the complementing tree in every $< \kappa$-generic extension. This is exactly Steel's proof of the weakly homogeneous case of the Martin–Solovay theorem. $\square$

We shall use Theorem 2.23 only in a localized form. Let M be one of the mice M_{n-1} and let δ be its least Woodin cardinal. If $T \in M$ is κ-weakly homogeneous in M for every $\kappa < \delta$, then for every forcing $\mathbb{P} \in M$ of M-cardinality $< \delta$, and every M-generic $G \subseteq \mathbb{P}$, the same tree T and its Martin–Solovay complement compute the same projective set in $M[G]$. Equivalently, for reals $x \in M[G]$, membership in $p[T]$ can be checked by the ground-model tree T and is not changed by passing to further forcing extensions of size below the relevant completeness bound.

Corollary 2.24 (Small generic absoluteness for a fixed weakly homogeneous tree)**.** *Let M be a transitive model containing a κ-weakly homogeneous tree T and a witnessing system $\bar{\mu}$. Let $S = \mathrm{ms}(\bar{\mu}, \Theta)$ for sufficiently large Θ. If G is generic over M for a forcing of M-cardinality $< \kappa$, then in $M[G]$,*

$$p[T] = \omega^\omega \backslash p[S].$$

In particular, if a projective formula $\varphi(x)$ is represented over M by the complementing pair (T, S), then for every real $x \in M[G]$,

$$M[G] \models \varphi(x) \quad \Longleftrightarrow \quad x \in p[T]^{M[G]}.$$

Proof. The first assertion is Theorem 2.23. For the final assertion, the projective formula is represented in M by the pair (T, S), and the pair remains an absolute complementing pair in $M[G]$. Thus exactly one of the trees has a branch over the real x in $M[G]$, and this is the same truth value assigned by the projective definition in the small generic extension. $\square$

Remark. Later forcing notions need not be regarded globally as small over the relevant mouse. What is used is the local consequence: every real and every name appearing in the coding construction is read in a bounded regular subforcing, and the relevant subforcing has size below the completeness bound of the homogeneity system. The assertion above is therefore applied in the intermediate model generated by that local support.

2.4 Defining the canonical trees T_n inside M_{n-1}

We now choose the particular trees to which the preceding subsection will be applied. Fix $n \geqslant 2$. Work in M_{n-1}, and let

$$\delta_0^n < \delta_1^n < \cdots < \delta_{n-2}^n$$

be the Woodin cardinals of M_{n-1}. We write simply δ_0 for the least of them when n is fixed.

Let $U_{n+1} \subseteq \omega^\omega$ be the standard universal $\mathbf{\Sigma}^1_{n+1}$ set, with the first real coordinate coding the index and the remaining coordinates coding the parameter and the argument. We fix this universal set once and for all using the usual Moschovakis parametrization conventions for projective pointclasses [9]. Thus every boldface $\mathbf{\Sigma}^1_{n+1}$ set is a section of U_{n+1}.

The following theorem is the precise object needed later. It is a standard consequence of the Martin–Steel analysis of projective scales, the Martin–Solovay construction, and Steel's tree-production argument for mice with finitely many Woodin cardinals [8, 13, 14].

Theorem 2.25 (Canonical weakly homogeneous tree for $\mathbf{\Sigma}^1_{n+1}$). *In M_{n-1} there is a tree*

$$T_n \subseteq (\omega \times \lambda_n)^{<\omega}$$

for some ordinal λ_n of M_{n-1} such that

$$p[T_n] = U_{n+1}$$

inside M_{n-1}. Moreover, for every $\kappa < \delta_0$, the tree T_n is κ-weakly homogeneous in M_{n-1}.

Proof. We recall the construction, since the parity shift is a common source of confusion. One begins with the projective pointclass immediately below $\mathbf{\Sigma}^1_{n+1}$. By the periodicity theorems and the Martin–Steel scale analysis, the relevant universal set at that lower level has a scale whose associated tree is homogeneous in M_{n-1}. This is the higher analogue of the familiar Martin–Solovay representation of the complete $\mathbf{\Sigma}^1_3$ set by the tree T_2 over M_1.

If the lower-level universal set occurs with the correct polarity, the tree of the scale already gives the required homogeneous representation. If the polarity is the complementary one, one applies the Martin–Solovay construction to the homogeneity system. The Martin–Solovay tree represents the complement by searching for coherent descending ordinal ranks in the corresponding ultrapowers. This is the step which accounts for the odd/even alternation in the projective hierarchy.

Finally, the passage from the lower-level matrix to the universal $\mathbf{\Sigma}^1_{n+1}$ set is an existential real projection. Homogeneous representations are stable under this operation in the weak sense: the additional real witness is absorbed into the weak choice of a countably complete tower. Equivalently, the projection of a homogeneously Suslin representation is weakly homogeneously Suslin. Thus one obtains a tree T_n with $p[T_n] = U_{n+1}$ and with κ-complete weak homogeneity systems for every $\kappa < \delta_0$.

All objects used in this construction are chosen inside M_{n-1}. The completeness of the systems below the least Woodin follows from the extender strength available in M_{n-1} and the standard Steel tree-production argument. □

For definiteness, we make the choice canonical.

Definition 2.26 (The canonical tree T_n)**.** *For $n \geqslant 2$, T_n denotes the $<_{M_{n-1}}$-least tree T for which M_{n-1} verifies the conclusion of Theorem 2.25. Along with T_n we fix the $<_{M_{n-1}}$-least coherent choice of weak homogeneity systems*

$$\bar{\mu}^n_\kappa \qquad (\kappa < \delta^n_0)$$

which witness that T_n is κ-weakly homogeneous. We also fix, for sufficiently large Θ_n, the Martin–Solovay complement

$$S_n = \mathrm{ms}(\bar{\mu}^n_\kappa, \Theta_n)$$

whenever a completeness bound κ has been specified. The value of S_n may depend on the chosen bound κ, but this will never matter: in each application we choose κ above the size of the relevant forcing.

Lemma 2.27 (Absoluteness of the canonical T_n representation)**.** *Let $n \geqslant 2$, let $\mathbb{P} \in M_{n-1}$ have M_{n-1}-cardinality $< \kappa < \delta^n_0$, and let $G \subseteq \mathbb{P}$ be M_{n-1}-generic. Then in $M_{n-1}[G]$ the ground-model tree T_n and the corresponding Martin–Solovay complement S_n are complements. Consequently, for every real $x \in M_{n-1}[G]$,*

$$M_{n-1}[G] \models x \in U_{n+1} \quad \Longleftrightarrow \quad x \in p[T_n]^{M_{n-1}[G]}.$$

The same equivalence remains true in all further forcing extensions of $M_{n-1}[G]$ by forcing notions of size below the remaining completeness bound.

Proof. By Definition 2.26, T_n is κ-weakly homogeneous in M_{n-1} via $\bar{\mu}^n_\kappa$. Theorem 2.23 gives a Martin–Solovay complement S_n such that T_n and S_n are κ-absolute complements. Since $|\mathbb{P}|^{M_{n-1}} < \kappa$, the complementing relation holds in $M_{n-1}[G]$.

The tree T_n represents the universal $\mathbf{\Sigma}^1_{n+1}$ set in the ground model, and the complementing pair remains absolute in the extension. Therefore a real x in the extension satisfies the projective universal formula exactly when the T_n-section above x is ill-founded. The final sentence is the same argument applied once more to the lifted homogeneity system. □

Corollary 2.28 (The case used in the main construction)**.** *In M_1 there is a canonical tree T_2 such that*

$$p[T_2] = U_3,$$

where U_3 is the universal $\mathbf{\Sigma}^1_3$ set. If δ is the Woodin cardinal of M_1, then for every $\kappa < \delta$, T_2 is κ-weakly homogeneous. Hence the T_2 representation of $\mathbf{\Sigma}^1_3$ truth is absolute to all forcing extensions of M_1 obtained by forcing of size $< \kappa$, and locally to all later intermediate extensions whose relevant support has size $< \kappa$.

Proof. This is Lemma 2.27 with $n = 2$. The tree T_2 is the Martin–Solovay tree in the form used by Hjorth and Solovay [7, 3]: it represents the universal $\boldsymbol{\Sigma}^1_3$ set and carries weak homogeneity systems below the Woodin cardinal of M_1. $\square$

Remark (how T_n will be used). The tree T_n is not an additional coding device. Its role is to make the lower projective truth predicates robust in small generic extensions. In the proof of Lebesgue measurability and the Baire property, the only instance needed is T_2: after a real parameter a appears, the bookkeeping adds Random and Cohen stages over bases containing $L[T_2, a]$, and amoeba stages over the same bases cover the null and meager Borel sets coded there. The weak homogeneity and Martin–Solovay complementing pair allow the $\boldsymbol{\Sigma}^1_3$ definition to be read correctly in those generic extensions.

3 The M_1-ground model and the Σ^1_4 coding apparatus

We now describe the ground model used for the detailed M_1-case and the coding predicate. The target value of the continuum in the final extension is $\aleph_2$, and the later bookkeeping iteration will have length ω_2. Since the preliminary branch forcing is c.c.c., this is the same ω_2 as in M_1.

3.1 The branch extension W

Using the canonical $\diamondsuit$-sequence of $M_1|\omega_1$, fix once and for all the canonical M_1-definable sequence

$$\vec{S} = \langle S_\xi \mid \xi < \omega_1 \rangle$$

of independent Suslin trees. Thus for every finite set $e \subseteq \omega_1$, the product

$$\prod_{\xi \in e} S_\xi$$

is again a Suslin tree. We identify each tree with a subset of ω_1 by the fixed M_1-definable coding of countable normal trees. This convention is used throughout the rest of the paper.

Let

$$Br = \prod_{\xi < \omega_1}^{\mathrm{fs}} S_\xi$$

be the finite-support branch product, computed in M_1. By independence of $\vec{S}$, every finite subproduct is Suslin, and the usual Δ-system argument shows that Br is c.c.c. If $H \subseteq Br$ is M_1-generic, write

$$b_\xi = \bigcup \{ p(\xi) : p \in H, \xi \in \mathrm{dom}(p) \}$$

for the cofinal branch through S_ξ added by the ξ-th coordinate. We set

$$W = M_1[H].$$

Thus W is just the extension obtained by adding a branch through every tree in the fixed sequence $\vec{S}$. Since Br has size ω_1 and is c.c.c., W preserves cardinals and satisfies $2^{\aleph_0} = \aleph_1$. The later length-ω_2 iteration will add $\aleph_2$ many reals and will be responsible for the final value $2^{\aleph_0} = \aleph_2$.

The point of having all branches present in W is that a coding forcing may use selected branches from H as parameters. The point of using the finite-support product is that every real in a local subextension depends on only countably many branch coordinates. This support feature will later allow us to compare the information decoded from a real with the branch coordinates which were actually used to produce that real.

3.2 Cohen coding areas

We fix the canonical M_1-definable almost disjoint family of reals

$$D = \langle d_\xi \mid \xi < \omega_1 \rangle.$$

It will be used at the last step of the coding, when a subset of ω_1 is almost-disjointly coded into a real.

We also fix the $<_{M_1}$-least bijection

$$\rho : ({}^{<\omega_1}2)^{M_1} \longrightarrow \omega_1$$

or, equivalently, the corresponding M_1-definable bijection between countable subsets of ω_1 and ω_1. Let $\mathbb{C}_{M_1}$ be the M_1-version of the σ-closed ω_1-Cohen forcing: conditions are countable binary sequences from M_1, ordered by end-extension. If $g \subseteq \omega_1$ is $\mathbb{C}_{M_1}$-generic, then every proper initial segment $g \restriction \alpha$ belongs to M_1, and we define the coding area determined by g by

$$C_g = \{\rho(g \restriction \alpha) \mid \alpha < \omega_1\} \subseteq \omega_1.$$

We shall say that $C \subseteq \omega_1$ is an M_1-Cohen coding area if $C = C_g$ for some such g. The standard fusion argument for $\mathbb{C}_{M_1}$ gives the two facts we need: each initial segment of C_g is computed in the relevant M_1-initial segment, and coding areas coming from mutually generic M_1-Cohen subsets of ω_1 are almost disjoint in that their intersection is countable.

Definition 3.1 (Selected branch coordinates)**.** *For $C \subseteq \omega_1$ and $u \in 2^\omega$, define*

$$\mathrm{Sel}(C, u) = \{\tau_u(\gamma, n) \mid \gamma \in C,\ n < \omega\},$$

where

$$\tau_u(\gamma, n) = \begin{cases} \omega \cdot \gamma + 2n, & \text{if } n \notin u, \\ \omega \cdot \gamma + 2n + 1, & \text{if } n \in u. \end{cases}$$

Thus $\mathrm{Sel}(C, u)$ *is the set of branch coordinates selected by the area* C *when it writes the real* u.

Lemma 3.2 (Countably many coding areas)**.** *Let* I *be countable. Let* $(C_i : i \in I)$ *be* M_1*-Cohen coding areas coming from mutually generic reservoir coordinates, and let* $u_i \in 2^\omega$ *for* $i \in I$. *Let* $B \in [\omega_1]^\omega$, *and put*

$$U = B \cup \bigcup_{i \in I} \mathrm{Sel}(C_i, u_i).$$

Then the following hold.

1. *If* $C \subseteq \omega_1$ *has size* ω_1 *and*
$$\forall i \in I \ (C \cap C_i \text{ is bounded in } \omega_1),$$
then there are ω_1*-many* $\gamma \in C$ *such that*
$$\{\omega \cdot \gamma + k \mid k < \omega\} \cap U = \varnothing.$$

2. *If* $i_0 \in I$, $C \cap C_{i_0}$ *is unbounded in* ω_1, *and* $w \in 2^\omega$ *satisfies* $w \neq u_{i_0}$, *then, for any* $n < \omega$ *with* $w(n) \neq u_{i_0}(n)$, *there are* ω_1*-many* $\gamma \in C \cap C_{i_0}$ *such that*
$$\tau_w(\gamma, n) \notin U.$$

Proof. For $i \neq j$ in I, the set $C_i \cap C_j$ is bounded in ω_1. Since I is countable, there is $\theta_0 < \omega_1$ such that

$$C_i \cap C_j \subseteq \theta_0 \qquad (i \neq j \text{ in } I).$$

Let

$$B^* = \{\gamma < \omega_1 \mid \exists k < \omega \ (\omega \cdot \gamma + k \in B)\}.$$

Then B^* is countable.

For (1), choose $\theta_1 < \omega_1$ such that

$$C \cap C_i \subseteq \theta_1 \qquad (i \in I).$$

Every $\gamma \in C \backslash (\theta_1 \cup B^*)$ has the required property: it does not lie in any C_i, and its ω-block does not meet B.

For (2), let $n < \omega$ be such that $w(n) \neq u_{i_0}(n)$. If

$$\gamma \in (C \cap C_{i_0}) \backslash (\theta_0 \cup B^*),$$

then $\gamma \notin C_i$ for all $i \neq i_0$, and the ω-block of γ does not meet B. At the area C_{i_0}, the coordinate selected by u_{i_0} at n is the other member of the pair

$$\{\omega \cdot \gamma + 2n, \ \omega \cdot \gamma + 2n + 1\}.$$

Hence $\tau_w(\gamma, n) \notin \mathrm{Sel}(C_{i_0}, u_{i_0})$, and it is not in any other $\mathrm{Sel}(C_i, u_i)$. Thus $\tau_w(\gamma, n) \notin U$. There are ω_1-many such γ. $\square$

3.3 Writing a real into the Suslin sequence

Let $w \in 2^\omega$ be a real in a generic extension of W. Suppose first that g is an M_1-Cohen subset of ω_1 and put $C = C_g$. For each $\gamma \in C$ we use the ω-block

$$\langle S_{\omega\cdot\gamma+k} \mid k < \omega\rangle$$

of the fixed sequence $\vec{S}$. The bit $w(n)$ is represented by choosing the branch through exactly one of the two trees

$$S_{\omega\cdot\gamma+2n}, \qquad S_{\omega\cdot\gamma+2n+1}.$$

Equivalently, in the notation fixed above, the selected coordinate is $\tau_w(\gamma, n)$. The branch used at the coordinate (γ, n) is $b_{\tau_w(\gamma,n)}$, already present in W. Thus any inner model which can correctly compute the trees $\vec{S}$ and which contains, for all $\gamma \in C$ and $n < \omega$, the selected branches

$$\langle b_{\tau_w(\gamma,n)} \mid \gamma \in C, n < \omega\rangle$$

can read the characteristic function of w from the pattern

$$\begin{aligned} n \in w &\iff S_{\omega\cdot\gamma+2n+1} \text{ has an } \omega_1\text{-branch},\\ n \notin w &\iff S_{\omega\cdot\gamma+2n} \text{ has an } \omega_1\text{-branch}, \end{aligned}$$

for every $\gamma \in C$.

To turn this branch pattern into a projective predicate, we do not code the branch data directly as a bare subset of ω_1. Instead we first package the relevant information into a transitive $\aleph_1$-sized model and then code that model. Let

$$\mathcal{B}_{g,w} = \langle b_{\tau_w(\gamma,n)} \mid \gamma \in C_g,\ n < \omega\rangle$$

be the indexed sequence of selected branches, coded as a set of triples $(\gamma, n, b_{\tau_w(\gamma,n)})$. Thus the domain of this sequence already determines C_g, but we shall keep C_g as an explicit object for readability.

Choose a transitive model

$$\mathcal{N} = \mathcal{N}_{g,w}$$

of size $\aleph_1$ such that

1. $\mathcal{N} \models \mathsf{ZFC}^- +$ "$\aleph_1$ exists" and $\omega_1^{\mathcal{N}} = \omega_1$;
2. $m_\infty = \mathcal{J}_{\omega_1}^{M_1}$ belongs to $\mathcal{N}$;
3. C_g and $\mathcal{B}_{g,w}$ belong to $\mathcal{N}$;
4. using m_∞, the model $\mathcal{N}$ computes the fixed sequence $\vec{S}$, the map ρ, and the almost disjoint family D, and it sees the branch pattern determined by $\mathcal{B}_{g,w}$ as coding the real w.

Here w is not used as a distinguished piece of the code. It is recovered from the selected branch pattern, while m_∞ supplies the definition of the Suslin sequence from which the pattern is read.

Fix a canonical code

$$X_{\mathcal{N}} = X_{g,w} \subseteq \omega_1$$

for the structure $(\mathcal{N}, \in, m_\infty, C_g, \mathcal{B}_{g,w})$. Concretely, choose the $<_{M_1}$-least enumeration of $\mathcal{N}$ of length ω_1 whose first distinguished entries are m_∞, C_g, and $\mathcal{B}_{g,w}$, and let $X_{\mathcal{N}}$ code the resulting well-founded extensional relation together with these distinguished constants. Any model which decodes $X_{\mathcal{N}}$ therefore recovers the transitive collapse $\mathcal{N}$ and the three named objects.

We now apply David's reshaping trick to $X_{\mathcal{N}}$. Choose an ordinal λ of size $\aleph_1$ such that

$$L_\lambda[X_{\mathcal{N}}] \models \mathsf{ZFC}^- + \text{“}\aleph_1 \text{ exists}''.$$

Inside $L_\lambda[X_{\mathcal{N}}]$, take a continuous increasing sequence

$$\langle M_\xi \mid \xi < \omega_1 \rangle$$

of countable elementary submodels, and put

$$c_\xi = M_\xi \cap \omega_1, \qquad C = \{c_\xi \mid \xi < \omega_1\}.$$

The reshaped set $Y = Y_{g,w} \subseteq \omega_1$ codes $X_{\mathcal{N}}$ on the odd ordinals and codes the club C on the even ordinals. More explicitly, the odd part of Y is a fixed copy of $X_{\mathcal{N}}$, while the even part codes the increasing enumeration of C by the usual interval pattern: before c_0 it codes a well-order of type c_0, between c_ξ and $c_{\xi+1}$ it codes a well-order of type $c_{\xi+1}$, and it is empty on the remaining intervals.

Consequently, if M is a countable transitive model of $\mathsf{ZFC}^- + \text{“}\aleph_1 \text{ exists}''$ and $Y \cap \omega_1^M \in M$, then M recognizes that $\omega_1^M \in C$. Inside its version of $L[Y \cap \omega_1^M]$, it recovers $X_{\mathcal{N}} \cap \omega_1^M$ and hence a local transitive model $\bar{\mathcal{N}}$ of size $\aleph_1^M$. This local model contains the corresponding initial segment m of M_1, the local coding area, and the local selected branch sequence. Since m defines the local version of $\vec{S}$, the model $\bar{\mathcal{N}}$ can read the same branch pattern and verify the local statement that the pattern codes w.

Finally, over the model containing Y, force with the Jensen–Solovay almost disjoint coding forcing

$$\mathbb{A}_D(Y)$$

relative to the fixed M_1-definable family D. If r_Y is the generic real, then

$$\alpha \in Y \iff r_Y \cap d_\alpha \text{ is finite}$$

for every $\alpha < \omega_1$, and this equivalence remains absolute to all outer models with the same interpretation of the initial segment of D.

For a real w we denote by

$$\mathrm{Code}(w)$$

the two-step forcing which first adds an M_1-Cohen subset g of ω_1 and then almost-disjointly codes the reshaped set $Y_{g,w}$ into a real:

$$\mathrm{Code}(w) = \mathbb{C}_{M_1} * \dot{\mathbb{A}}_D(\dot{Y}_{\dot{g},w}).$$

When w is a name over an initial segment of the later iteration, the same formula is read as the corresponding name for this two-step forcing.

Later, once hybrid presentations have been introduced, we use the same notation $\mathrm{Code}(\dot{u})$ for the hybrid coding operation associated with this two-step forcing. Thus the $\mathbb{C}_{M_1}$-factor is inserted as a fresh reservoir product coordinate, while the almost-disjoint forcing $\dot{\mathbb{A}}_D(\dot{Y}_{\dot{g},\dot{u}})$ is inserted as the corresponding real-adding coordinate. In other words, $\mathrm{Code}(\dot{u})$ denotes an ordinary two-step forcing in the local description above, but denotes this product/iteration insertion when it occurs inside a hybrid presentation.

3.4 The predicates LocalCode, Ψ, and Φ

We isolate the projective content of the preceding construction. The notation

$$M \models \mathrm{LocalCode}(Y, w, m)$$

will be used for the following first-order assertion over a transitive model M of $\mathsf{ZFC}^- +$ "$\aleph_1$ exists", with $m \in M$ as a displayed parameter. The set $Y \subseteq \omega_1^M$ is a reshaped David code: its odd part decodes a set $X_{\mathcal{N}} \subseteq \omega_1^M$, its even part supplies the reshaping club, and $X_{\mathcal{N}}$ codes a well-founded extensional structure whose transitive collapse is a model

$$\bar{\mathcal{N}}$$

of size $\aleph_1^M$ such that

- $\bar{\mathcal{N}} \models \mathsf{ZFC}^- +$ "$\aleph_1$ exists";
- $\bar{\mathcal{N}}$ contains the distinguished initial segment $m \in \mathcal{I}$ with $\omega_1^m = \omega_1^M$;
- using m, the model $\bar{\mathcal{N}}$ computes the local versions of $\vec{S}$, ρ, and the almost disjoint family D;
- $\bar{\mathcal{N}}$ contains an M_1-Cohen coding area C computed from m, i.e. a set of the form
$$C = \{\rho^m(g \upharpoonright \alpha) \mid \alpha < \omega_1^m\}$$
for the M_1-Cohen subset $g \subseteq \omega_1^m$ decoded in $\bar{\mathcal{N}}$, and it contains an indexed sequence of selected branches
$$\mathcal{B} = \langle c_{\gamma,n} \mid \gamma \in C, n < \omega\rangle;$$

- the branch sequence verifies, in $\bar{\mathcal{N}}$, the pattern coding w:

$$\begin{aligned} n \in w &\iff \bar{\mathcal{N}} \models \text{“}S^m_{\omega\cdot\gamma+2n+1} \text{ has an } \omega_1^m\text{-branch''}, \\ n \notin w &\iff \bar{\mathcal{N}} \models \text{“}S^m_{\omega\cdot\gamma+2n} \text{ has an } \omega_1^m\text{-branch''} \end{aligned}$$

  for every $\gamma \in C$ and every $n < \omega$.

Here S^m_ξ denotes the ξ-th tree of the sequence computed from m. Thus Y does not merely list branches; it codes a local universe in which the relevant M_1-initial segment defines the Suslin sequence, and the selected branches then determine the real w.

Let $D^m = \langle d^m_\alpha \mid \alpha < \omega_1^m \rangle$ denote the canonical almost disjoint family as computed in m. We define $\Psi(r, w)$ by the following formula:

$$\begin{aligned} \Psi(r,w) \quad \iff \quad \forall M\, \forall m \Big(& M \text{ is a countable transitive model of } \mathsf{ZFC}^- + \text{“}\aleph_1 \text{ exists''} \\ & \wedge\ r, w, m \in M \\ & \wedge\ m \in \mathcal{I} \\ & \wedge\ \omega_1^m = \omega_1^M \\ & \Longrightarrow \exists Y \in M \Big[Y \subseteq \omega_1^M \\ & \qquad \wedge\ \forall \alpha < \omega_1^M\ (\alpha \in Y \iff r \cap d^m_\alpha \text{ is finite}) \\ & \qquad \wedge\ M \models \mathrm{LocalCode}(Y, w, m) \Big] \Big). \end{aligned}$$

Thus $\Psi(r, w)$ says that every relevant countable transitive model which has the correct M_1-initial segment at its ω_1 decodes from r, using the local almost disjoint family, a reshaped code for a local transitive model whose branch sequence codes w.

The assertion $\Psi(r, w)$ is a $\mathbf{\Pi}^1_3$ statement. The only non-arithmetic ingredient is the recognition of the correct M_1-initial segments, which is $\mathbf{\Pi}^1_2$ via the class $\mathcal{I}$; the remaining decoding and branch-pattern checks are absolute first-order checks inside the countable transitive model. Therefore

$$\Phi(w) \quad \iff \quad \exists r\, \Psi(r, w)$$

is a $\mathbf{\Sigma}^1_4$ predicate. We read $\Phi(w)$ as “w is coded into the fixed Suslin sequence $\vec{S}$”.

Lemma 3.3 (The inner model decoded by a Φ-witness)**.** *Assume that r witnesses $\Phi(w)$, that is, assume that $\Psi(r, w)$ holds. Put*

$$m_\infty = \bigcup \mathcal{I} = \mathcal{J}^{M_1}_{\omega_1}.$$

Then the inner model $L[r, m_\infty]$ contains the real w and contains the objects decoded from r: a reshaped set $Y_r \subseteq \omega_1$, a model code $X_{\mathcal{N},r} \subseteq \omega_1$, the *transitive model $\mathcal{N}_r$ decoded from it, an M_1-Cohen coding area C_r, and the selected branch sequence through the trees of the fixed sequence $\vec{S}$. Moreover $L[r, m_\infty]$ sees the branch pattern coding w; more explicitly, in $L[r, m_\infty]$ the decoded objects satisfy*

$$\forall\gamma \in C_r \, \forall n < \omega \quad n \in w \iff S_{\omega\cdot\gamma+2n+1} \text{ has an } \omega_1\text{-branch},$$
$$\forall\gamma \in C_r \, \forall n < \omega \quad n \notin w \iff S_{\omega\cdot\gamma+2n} \text{ has an } \omega_1\text{-branch}.$$

Thus a witness r for $\Phi(w)$ does not merely certify the projective statement externally; together with $\mathcal{J}^{M_1}_{\omega_1}$ it generates an actual inner model in which the w-pattern is present.

Proof. All decoding procedures used below are the fixed canonical ones. In particular, once r and an initial segment $m \in \mathcal{I}$ are given, the almost-disjoint decoding below ω_1^m is unique.

First $L[r, m_\infty]$ can form the almost-disjoint decoding of r with respect to the canonical family computed from m_∞:

$$Y_r = \{\alpha < \omega_1 \mid r \cap d^{m_\infty}_\alpha \text{ is finite}\}.$$

This set belongs to $L[r, m_\infty]$ and is uniquely determined by r and m_∞.

We claim that the David decoding of Y_r in $L[r, m_\infty]$ succeeds and produces the asserted model code and branch data. Suppose not. Then some finite part of the canonical decoding fails: either the reshaping test fails, or the odd part does not decode a well-founded extensional structure of the required kind, or the resulting collapse does not contain the required distinguished objects, or one of the branch-pattern clauses fails. Choose a sufficiently large regular Θ and take, in the ambient universe, a countable elementary submodel

$$X \prec H_\Theta$$

containing r, w, m_∞, and a witness to this failure. Let

$$\pi : X \longrightarrow \bar{M}$$

be the transitive collapse, and put $m = \pi(m_\infty)$. By the recovery and condensation facts for the cofinal system $\mathcal{I}$, the model m is a member of $\mathcal{I}$ and

$$\omega_1^m = \omega_1^{\bar{M}}.$$

Moreover $\bar{M}$ is a countable transitive model of $\mathsf{ZFC}^- +$ "$\aleph_1$ exists", and $r, w, m \in \bar{M}$.

Inside $\bar{M}$, the almost-disjoint decoding of r with respect to D^m is exactly

$$Y_r \cap \omega_1^{\bar{M}}.$$

Indeed, for every $\alpha < \omega_1^{\bar{M}}$, the real d^m_α is the corresponding initial member of the canonical almost disjoint family computed from m_∞, and hence

$$\alpha \in Y_r \cap \omega_1^{\bar{M}} \quad \Longleftrightarrow \quad r \cap d^m_\alpha \text{ is finite.}$$

The reshaping and odd/even decoding are first-order procedures over the relevant transitive model, so the failure chosen above reflects to $\bar{M}$. Thus

$$\bar{M} \not\models \mathrm{LocalCode}(Y_r \cap \omega_1^{\bar{M}}, w, m).$$

But $\Psi(r, w)$ applies to the countable transitive model $\bar{M}$ and the initial segment m. It therefore yields some $Y \in \bar{M}$ such that

$$\forall \alpha < \omega_1^{\bar{M}} \left(\alpha \in Y \Longleftrightarrow r \cap d^m_\alpha \text{ is finite}\right)$$

and

$$\bar{M} \models \mathrm{LocalCode}(Y, w, m).$$

By uniqueness of the almost-disjoint decoding, $Y = Y_r \cap \omega_1^{\bar{M}}$, a contradiction. Hence the global decoding in $L[r, m_\infty]$ succeeds. Let $X_{\mathcal{N},r}$ be the model code decoded from the odd part of Y_r, let $\mathcal{N}_r$ be the transitive collapse of the structure coded by $X_{\mathcal{N},r}$, and let C_r and the selected branch sequence be the distinguished objects decoded inside $\mathcal{N}_r$.

It remains to see that the real read from this branch pattern is the original real w. Since the global decoding has succeeded, $L[r, m_\infty]$ defines a real w^* by reading the decoded branches: for $n < \omega$, put $n \in w^*$ iff, equivalently for every $\gamma \in C_r$, the decoded branch sequence witnesses that

$$S_{\omega\cdot\gamma+2n+1} \text{ has an } \omega_1\text{-branch.}$$

The local-code clauses ensure that this definition is independent of the choice of $\gamma \in C_r$.

Suppose toward a contradiction that $w^* \neq w$, and choose $n < \omega$ such that $w^*(n) \neq w(n)$. Again take a sufficiently large countable elementary submodel $X \prec H_\Theta$ in the ambient universe containing r, w, m_∞, n and enough of the decoded global data to witness the value of $w^*(n)$, and collapse it to $\bar{M}$, with $m = \pi(m_\infty)$. As above, $\bar{M}$ is a relevant countable transitive model, $m \in \mathcal{I}$, $\omega_1^m = \omega_1^{\bar{M}}$, and the almost-disjoint decoding of r in $\bar{M}$ is $Y_r \cap \omega_1^{\bar{M}}$. By elementarity and the absoluteness of the decoding procedure, $\bar{M}$ reads the n-th bit of the local branch pattern as $w^*(n)$. On the other hand, $\Psi(r, w)$ says that the same uniquely decoded set satisfies

$$\bar{M} \models \mathrm{LocalCode}(Y_r \cap \omega_1^{\bar{M}}, w, m),$$

so $\bar{M}$ reads the n-th bit of that same branch pattern as $w(n)$. This contradicts $w^*(n) \neq w(n)$.

Therefore $w^* = w$. Since w^* is definable in $L[r, m_\infty]$ from the decoded objects, we have $w \in L[r, m_\infty]$. The decoded objects also belong to $L[r, m_\infty]$ by construction, and the displayed branch-pattern equivalences hold there. This proves the lemma. $\square$

3.5 Localized presentations of names

We now fix the presentation convention which will be used throughout the construction. Recall that the first forcing over M_1 is the finite-support branch product

$$Br = \prod_{\xi<\omega_1}^{\mathrm{fs}} S_\xi,$$

which adds a cofinal branch through every tree in the fixed independent sequence $\vec{S}$. If $H \subseteq Br$ is M_1-generic, we write

$$W = M_1[H].$$

All later forcing is described over this branch extension. A local hybrid presentation over W will be denoted by $\mathcal{R}$, and its initial presentation up to stage β by $\mathcal{R}_\beta$. Once the presentation is fixed, $\mathbb{P}_\beta$ denotes the post-branch forcing over W associated with $\mathcal{R}_\beta$. If $\mathcal{R} = \mathcal{R}_\delta$ is a completed presentation, then $\mathbb{P} = \mathbb{P}_\delta$ denotes its terminal post-branch forcing.

The corresponding full forcing over M_1, including the initial branch block, is written

$$Br \ltimes_{\mathrm{hyb}} \mathcal{R}_\beta.$$

Here $Br \ltimes_{\mathrm{hyb}} \mathcal{R}_\beta$ means: first force with Br, form $W = M_1[H]$, and then force over W with the post-branch forcing $\mathbb{P}_\beta$ associated with $\mathcal{R}_\beta$. The symbol $\ltimes_{\mathrm{hyb}}$ is only notation for this composition with a post-branch hybrid presentation; it is not an ordinary iteration symbol. Equivalently, if $H \subseteq Br$ is generic, then the quotient of $Br \ltimes_{\mathrm{hyb}} \mathcal{R}_\beta$ over H is canonically isomorphic to $\mathbb{P}_\beta$.

We also fix the convention for the forcings which deal with Lebesgue measure and the Baire property; we call them the *regularity forcings.* If W' is a transitive inner model of the current ambient universe, with the relevant ω_1 and enough set theory to compute Borel codes, measure, category, and the corresponding forcing notions, then

$$\mathbb{B}_{\mathrm{rand}}^{W'}$$

denotes the Random algebra as computed in W'. Its conditions are the positive Borel sets coded in W', modulo null equivalence as computed in W'. We also write

$$\mathbb{A}_{\mathcal{N}}^{W'} \qquad \text{and} \qquad \mathbb{A}_{\mathcal{M}}^{W'}$$

for the standard amoeba forcing for the null ideal, respectively for the meager ideal, as computed in W'. Thus $\mathbb{A}_{\mathcal{N}}^{W'}$ adds a Borel null set covering every Borel null set coded in W', and $\mathbb{A}_{\mathcal{M}}^{W'}$ adds a Borel meager set covering every Borel meager set coded in W'. We use fixed Borel-code presentations of these amoeba forcings in which they are σ-linked. In particular, these forcings preserve Suslin trees from the ground model.

The only bases used for regularity stages are finite-real-parameter bases of the form

$$L[T_2, a_0, \dots, a_{k-1}].$$

Here $T_2 \in M_1$ is the fixed Martin–Solovay tree from Subsection 2.4, and the reals $a_0, \dots, a_{k-1}$ are read from the current generic extension. When a regularity forcing computed in such a base is used inside a larger ambient extension, it is not recomputed in the larger universe. The conditions and the order are the ones computed in the displayed base.

The post-branch forcing is a hybrid of a product and an iteration. We now give its official syntactic form. A *resolved hybrid bookkeeping* of length β is a sequence

$$F = \langle F(\eta) \mid \eta < \beta \rangle$$

whose η-th value is decoded over the already constructed initial forcing as one of the following tags:

$$\mathbf{1}, \qquad \mathrm{Coh}, \qquad \mathrm{Code}(\dot{u}_\eta),$$

or one of the relative regularity tags

$$(\mathrm{Rand}; \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1}), \qquad (\mathrm{Am}_{\mathcal{N}}; \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1}), \qquad (\mathrm{Am}_{\mathcal{M}}; \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1}).$$

Here $k_\eta < \omega$, $\dot{u}_\eta$ is a name for a real over the current initial forcing, and each $\dot{a}^\eta_i$ is a name for a real over the current initial forcing. The word "resolved" is important: this bookkeeping tells the hybrid presentation what forcing to use next. The later allowability recursions may start from higher-level bookkeeping data, such as a well-order pair or a uniformization tuple, but before a hybrid stage is formed that data is resolved into one of the tags listed above.

Given such an F, we recursively build a post-branch hybrid presentation

$$\langle \mathcal{R}_\eta \mid \eta \leqslant \beta \rangle$$

together with its associated post-branch forcings $\langle \mathbb{P}_\eta \mid \eta \leqslant \beta \rangle$. We put $\mathbb{P}_0 = \mathbf{1}$. At a successor stage $\eta + 1$, after $\mathbb{P}_\eta$ has been constructed, we first add a fresh reservoir coordinate $\mathbb{C}_\eta$, a copy of the M_1-computed ω_1-Cohen forcing. Thus the preliminary forcing at this stage is

$$\widehat{\mathbb{P}}_{\eta+1} = \mathbb{P}_\eta \times \mathbb{C}_\eta,$$

where $\dot{g}_\eta$ denotes the $\mathbb{C}_\eta$-generic. The second-step forcing $\dot{\mathbb{Q}}_\eta$ over $\widehat{\mathbb{P}}_{\eta+1}$ is determined by the decoded tag $F(\eta)$:

1. If the tag is $\mathbf{1}$, then $\dot{\mathbb{Q}}_\eta$ is the trivial forcing.
2. If the tag is Coh, then $\dot{\mathbb{Q}}_\eta$ is ordinary Cohen forcing.

3. If the tag is $\mathrm{Code}(\dot{u}_\eta)$, then

$$\dot{\mathbb{Q}}_\eta = \dot{\mathbb{A}}_D(\dot{Y}_{\dot{g}_\eta, \dot{u}_\eta}).$$

   This is the Jensen–Solovay almost-disjoint coding forcing which codes the reshaped set determined by the fresh reservoir generic and the real named by $\dot{u}_\eta$.

4. If the tag is $(\mathrm{Rand}; \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1})$, put

$$\dot{W}_\eta = L[T_2, \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1}],$$

   with all real names lifted to $\widehat{\mathbb{P}}_{\eta+1}$. The second-step forcing is

$$\dot{\mathbb{Q}}_\eta = \dot{\mathbb{B}}^{\dot{W}_\eta}_{\mathrm{rand}},$$

   the Random algebra computed in the displayed base.

5. If the tag is $(\mathrm{Am}_{\mathcal{N}}; \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1})$, respectively $(\mathrm{Am}_{\mathcal{M}}; \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1})$, put

$$\dot{W}_\eta = L[T_2, \dot{a}^\eta_0, \dots, \dot{a}^\eta_{k_\eta - 1}],$$

   again with all real names lifted to $\widehat{\mathbb{P}}_{\eta+1}$. The second-step forcing is the measure-amoeba, respectively category-amoeba, forcing computed in $\dot{W}_\eta$.

Finally we set

$$\mathbb{P}_{\eta+1} = \widehat{\mathbb{P}}_{\eta+1} * \dot{\mathbb{Q}}_\eta.$$

At limit stages we take the mixed-support limit: countable support on the reservoir product coordinates and finite support on the c.c.c. second-step coordinates. Thus every successor stage carries a fresh reservoir coordinate, but Φ reads such a coordinate only when the second step is an almost-disjoint coding forcing. We call the resulting object a *local hybrid presentation over* W.

We shall compare names by means of canonical localized presentations. Since $W = M_1[H]$, we use the fixed M_1-coding of W-objects by Br-names. Whenever an object of W is used as a parameter, it is represented by its $<_{M_1}$-least Br-name. Thus the collection of codes for the names and supports below is ordered by the canonical M_1-construction order.

Definition 3.4 (Admissible supports and canonical lifts)**.** *Let $\mathcal{R}_\beta$ be a local hybrid presentation over W, with resolved bookkeeping F, and let $A \subseteq \beta$. We define, by induction on $\eta \leqslant \beta$, the following objects simultaneously:*

1. *the assertion that $A \cap \eta$ is admissible for $\mathcal{R}_\eta$;*
2. *the restricted hybrid presentation $\mathcal{R}_{A\cap\eta}$ and its forcing $\mathbb{P}_{A\cap\eta}$;*

3. *the canonical complete embedding*

$$i^{\eta}_{A} : \mathbb{P}_{A\cap\eta} \longrightarrow \mathbb{P}_{\eta};$$

4. *the canonical lift of names along this embedding.*

At $\eta = 0$, the restricted presentation is trivial and the embedding is the identity.

At a limit $\lambda \leqslant \beta$, $A \cap \lambda$ is admissible if $A \cap \eta$ is admissible for every $\eta < \lambda$ and the restricted presentations are coherent. In that case $\mathcal{R}_{A\cap\lambda}$ is the mixed-support limit of the earlier restricted presentations, and i^{λ}_{A} is the coordinatewise direct limit of the embeddings i^{η}_{A}, $\eta < \lambda$.

Suppose now that $\eta < \beta$ and that $A \cap \eta$ has already been declared admissible. If $\eta \notin A$, then the restricted presentation skips the η-th stage:

$$\mathbb{P}_{A\cap(\eta+1)} = \mathbb{P}_{A\cap\eta}.$$

The embedding $i^{\eta+1}_{A}$ extends i^{η}_{A} by placing the trivial condition at the fresh reservoir coordinate $\mathbb{C}_{\eta}$ and at the second-step coordinate $\dot{\mathbb{Q}}_{\eta}$.

If $\eta \in A$, then the η-th stage must be readable over the earlier restricted forcing. We first keep the fresh reservoir coordinate and form

$$\widehat{\mathbb{P}}_{A,\eta+1} = \mathbb{P}_{A\cap\eta} \times \mathbb{C}_{\eta},$$

with the embedding induced by $i^{\eta}_{A} \times \mathrm{id}_{\mathbb{C}_{\eta}}$. The second-step forcing is then defined according to the resolved tag $F(\eta)$:

1. *For the tags* **1** *and* Coh, *the restricted stage uses the same trivial, respectively ordinary Cohen, second-step forcing.*

2. *If $F(\eta) = \mathrm{Code}(\dot{u}_{\eta})$, then there must be a $\mathbb{P}_{A\cap\eta}$-name $\dot{u}^{A}_{\eta}$ such that*

$$\mathbb{P}_{\eta} \Vdash \dot{u}_{\eta} = (\dot{u}^{A}_{\eta})^{\uparrow\eta}.$$

   If no such name exists, then $A \cap (\eta + 1)$ is not admissible. If such a name exists, we use the $<_{M_1}$-least one and put

$$\dot{\mathbb{Q}}^{A}_{\eta} = \dot{\mathbb{A}}_{D}(\dot{Y}_{\dot{g}_{\eta},\dot{u}^{A}_{\eta}})$$

   over $\widehat{\mathbb{P}}_{A,\eta+1}$.

3. *If $F(\eta)$ is a relative regularity tag*

$$(\mathrm{Rand}; \dot{a}^{\eta}_{0}, \dots, \dot{a}^{\eta}_{k_{\eta}-1}), \quad (\mathrm{Am}_{\mathcal{N}}; \dot{a}^{\eta}_{0}, \dots, \dot{a}^{\eta}_{k_{\eta}-1}), \quad \textit{or} \quad (\mathrm{Am}_{\mathcal{M}}; \dot{a}^{\eta}_{0}, \dots, \dot{a}^{\eta}_{k_{\eta}-1}),$$

   then for every $i < k_{\eta}$ there must be a $\mathbb{P}_{A\cap\eta}$-name $\dot{a}^{\eta,A}_{i}$ such that

$$\mathbb{P}_{\eta} \Vdash \dot{a}^{\eta}_{i} = (\dot{a}^{\eta,A}_{i})^{\uparrow\eta}.$$

If one of these names does not exist, then $A \cap (\eta + 1)$ *is not admissible. If they exist, use the* $<_{M_1}$*-least such tuple and let*

$$\dot{W}^A_\eta = L[T_2, \dot{a}^{\eta,A}_0, \ldots, \dot{a}^{\eta,A}_{k_\eta - 1}].$$

The restricted stage uses the same kind of regularity forcing computed in $\dot{W}^A_\eta$*: Random, measure-amoeba, or category-amoeba according to the tag.*

If the relevant requirement is satisfied, then

$$\mathbb{P}_{A\cap(\eta+1)} = \widehat{\mathbb{P}}_{A,\eta+1} * \dot{\mathbb{Q}}^A_\eta,$$

and the embedding $i^{\eta+1}_A$ *is the natural two-step extension of* i^η_A*.*

Once i^η_A *has been defined, the canonical lift of a* $\mathbb{P}_{A\cap\eta}$*-name* σ *to a* $\mathbb{P}_\eta$*-name is the recursive name translation*

$$\sigma^{\uparrow\eta} = \{(\tau^{\uparrow\eta}, i^\eta_A(p)) \mid (\tau, p) \in \sigma\}.$$

When $\eta = \beta$ *and* $A \cap \beta = A$ *is admissible, we simply write* $\mathbb{P}_A$*,* i^β_A*, and* $\sigma^{\uparrow\beta}$*.*

Thus, if $G_\beta \subseteq \mathbb{P}_\beta$ is generic and $G_A = (i^\beta_A)^{-1}[G_\beta]$, then G_A is $\mathbb{P}_A$-generic and

$$(\sigma^{\uparrow\beta})^{G_\beta} = \sigma^{G_A}$$

for every $\mathbb{P}_A$-name σ. The embedding i^β_A identifies $\mathbb{P}_A$ with a complete subforcing of $\mathbb{P}_\beta$. In what follows we suppress the embedding and regard $\mathbb{P}_A$ as this complete subforcing.

If $\vec{\tau}$ is a finite tuple of $\mathbb{P}_\beta$-names for reals or natural numbers, then an admissible support $A \subseteq \beta$ is *admissible for* $\vec{\tau}$ if every member of $\vec{\tau}$ is the canonical lift of a $\mathbb{P}_A$-name. There may be many admissible supports for $\vec{\tau}$, and there need not be a unique inclusion-minimal one. We therefore set

$$\mathrm{supp}_{\mathrm{loc}}(\vec{\tau}, \mathbb{P}_\beta) := \text{the } <_{M_1} \text{-least admissible support for } \vec{\tau}.$$

The notation

$$\mathbb{P}_{\mathrm{supp}_{\mathrm{loc}}(\vec{\tau},\mathbb{P}_\beta)}$$

always means the complete subforcing determined by this least admissible support. The definition of $\mathrm{supp}_{\mathrm{loc}}$ has no size clause. Countable localization is proved later for the presentations which come from the allowable recursion.

Definition 3.5 (Localized presentation)**.** *Let* $\dot{x}$ *be a* $\mathbb{P}_\beta$*-name for a real. A* localized presentation *of* $\dot{x}$ *is a pair* (A, σ) *such that* $A \subseteq \beta$ *is an admissible support,* σ *is a* $\mathbb{P}_A$*-name for a real, and*

$$1_{\mathbb{P}_\beta} \Vdash \dot{x} = \sigma^{\uparrow\beta}.$$

We order localized presentations by the canonical $<_{M_1}$*-order of their codes. The* canonical localized presentation *of* $\dot{x}$ *is the* $<_{M_1}$*-least localized presentation of* $\dot{x}$*.*

If $G_\beta \subseteq \mathbb{P}_\beta$ is generic and (A_i, σ_i) is the canonical localized presentation of $\dot{z}_i$, then

$$z_i = \dot{z}_i^{G_\beta} = \sigma_i^{G_{A_i}},$$

where G_{A_i} is the induced $\mathbb{P}_{A_i}$-generic filter. For two names $\dot{z}_0, \dot{z}_1$ we write

$$\dot{z}_0 \lhd_{\mathrm{loc}} \dot{z}_1$$

if the canonical localized presentation of $\dot{z}_0$ is $<_{M_1}$-below the canonical localized presentation of $\dot{z}_1$. This is the comparison used by the well-order coding stages below.

Definition 3.6 (Regularity-stage supports and branch footprints)**.** *Let $\mathcal{R}_\beta$ be a local hybrid presentation over $W = M_1[H]$, and let $\eta < \beta$ be a relative regularity stage. Suppose that the resolved tag at η is one of*

$$(\mathrm{Rand}; \dot{a}_0^\eta, \ldots, \dot{a}_{k_\eta - 1}^\eta), \qquad (\mathrm{Am}_{\mathcal{N}}; \dot{a}_0^\eta, \ldots, \dot{a}_{k_\eta - 1}^\eta), \qquad (\mathrm{Am}_{\mathcal{M}}; \dot{a}_0^\eta, \ldots, \dot{a}_{k_\eta - 1}^\eta).$$

The base used at this stage is, by definition,

$$\dot{W}_\eta = L[T_2, \dot{a}_0^\eta, \ldots, \dot{a}_{k_\eta - 1}^\eta],$$

with the evident meaning $L[T_2]$ when $k_\eta = 0$. The regularity forcing at the stage is the Random algebra, measure-amoeba forcing, or category-amoeba forcing computed in this displayed base, according to the tag.

A support witness *for the regularity stage is a tuple*

$$(A_{\mathrm{rb}}, \langle \dot{a}_i^{\mathrm{rb}} \mid i < k_\eta \rangle, B_{\mathrm{rb}}^0, \dot{B}_{\mathrm{rb}})$$

such that:

1. *$A_{\mathrm{rb}} \subseteq \eta$ is a countable admissible support, and each $\dot{a}_i^{\mathrm{rb}}$ is a $\mathbb{P}_{A_{\mathrm{rb}}}$-name for a real;*
2. *for every $i < k_\eta$,*
$$\mathbb{P}_\eta \Vdash \dot{a}_i^\eta = (\dot{a}_i^{\mathrm{rb}})^{\uparrow\eta};$$
3. *every object of $W = M_1[H]$ occurring in the codes of the names $\dot{a}_i^{\mathrm{rb}}$ is represented by its $<_{M_1}$-least Br-name, and the union of the supports of these Br-names is contained in the countable set $B_{\mathrm{rb}}^0 \subseteq \omega_1$.*

Let

$$\dot{W}_{\mathrm{rb}} = L[T_2, \dot{a}_0^{\mathrm{rb}}, \ldots, \dot{a}_{k_\eta - 1}^{\mathrm{rb}}]$$

be the base read on the support A_{rb}. Thus the canonical lift of $\dot{W}_{\mathrm{rb}}$ to stage η is forced to be the stage base $\dot{W}_\eta$.

Using the notation from Definition 3.1, let $E(A_{\mathrm{rb}})$ be the set of explicit coding coordinates in A_{rb}. If $\nu \in E(A_{\mathrm{rb}})$, let $\dot{C}_\nu$ be the $\mathbb{P}_{A_{\mathrm{rb}}}$-name for the *coding area of the reservoir generic at* ν*, and let* $\dot{u}_\nu$ *be the* $\mathbb{P}_{A_{\mathrm{rb}}}$*-name coded at* ν*. The branch footprint of the witness is the* $\mathbb{P}_{A_{\mathrm{rb}}}$*-name*

$$\dot{B}_{\mathrm{rb}} = \check{B}^0_{\mathrm{rb}} \cup \bigcup_{\nu \in E(A_{\mathrm{rb}})} \mathrm{Sel}(\dot{C}_\nu, \dot{u}_\nu).$$

Thus $\dot{B}_{\mathrm{rb}}$ *is generated by countably many coding areas and a countable set of branch coordinates. It may have size* ω_1*.*

Among all support witnesses for the stage, choose the $<_{M_1}$*-least one. We call the associated quadruple*

$$(A^{\mathrm{reg}}_\eta, \dot{W}^{\mathrm{reg}}_\eta, B^{0,\mathrm{reg}}_\eta, \dot{B}^{\mathrm{reg}}_\eta)$$

the canonical support witness *for the regularity stage, where*

$$\dot{W}^{\mathrm{reg}}_\eta = L[T_2, \dot{a}^{\mathrm{reg}}_0, \ldots, \dot{a}^{\mathrm{reg}}_{k_\eta - 1}]$$

is the base read on the least support witness. The regularity forcing is always computed in the relevant $L[T_2, \vec{a}]$*-base. It is not recomputed after passing to a larger ambient extension.*

Remark 3.7 (Regularity bases and omission)**.** *Definition 3.6 records the support and branch-footprint data which keep the regularity stages usable in omitting models. If a stage used Random or amoeba forcing computed over the whole branch extension* $W = M_1[H]$*, then the stage would not in general be interpretable over*

$$M_1[H \upharpoonright (\omega_1 \backslash \{\xi\})].$$

In the definition above the base is instead generated from finitely many real parameters on a countable admissible support:

$$L[T_2, a_0, \ldots, a_{k-1}].$$

The tree T_2 *is fixed ground-model data and contributes no branch-product coordinate. The branch-product coordinates relevant to the base are precisely those used by the* W*-parameters occurring in the names for the reals* a_i*, together with the coordinates generated by explicit coding stages in the admissible support on which those names are read. This is recorded by the name* $\dot{B}^{\mathrm{reg}}_\eta$*.*

The branch footprint of such a base is not required to be countable; it is generated by countably many coding areas and countably many ground branch coordinates. Its complement has size ω_1*: above one bound, each block meets at most one generating coding area, and from each pair* $\{\omega\cdot\gamma+2n, \omega\cdot\gamma+2n+1\}$ *at most one coordinate is selected. This is the support form used in the branch omission and no-unwanted-code arguments.*

Thus, for a real parameter a *with a countable local presentation,* $L[T_2, a]$ *is a base of the allowed form, and similarly for any finite tuple of real parameters read on a countable admissible support. The bookkeeping inserts relative Random, relative measure-amoeba, and relative category-amoeba stages over such bases cofinally often.*

3.6 Allowable and α-allowable local coding iterations

We now formulate the forcing recursion using the localized-presentation convention. Fix, once and for all, a universal lightface sequence

$$\langle A_m \mid m < \omega \rangle$$

of Π^1_4 subsets of $(2^\omega)^2$ in the M_1-case. Real parameters are handled later by the standard pairing argument, i.e. by applying the lightface list to relations in the paired variable (a, x). In the higher M_n-case this sequence is replaced by the corresponding universal lightface sequence of Π^1_{n+3} sets. A bookkeeping function is an element of M_1. Its values are M_1-descriptions which are decoded relative to the current associated post-branch forcing $\mathbb{P}_\beta$. We suppress the decoding map and write, for example,

$$F(\beta) = (\mathrm{Rand}; \dot{a}_0, \ldots, \dot{a}_{k-1}), \qquad F(\beta) = \dot{w}, \qquad F(\beta) = (\dot{z}_0, \dot{z}_1), \qquad F(\beta) = (\dot{x}, \dot{z}, \dot{m}),$$

when the decoded value has the corresponding form. The regularity tags may also be

$$(\mathrm{Am}_{\mathcal{N}}; \dot{a}_0, \ldots, \dot{a}_{k-1}) \quad \text{or} \quad (\mathrm{Am}_{\mathcal{M}}; \dot{a}_0, \ldots, \dot{a}_{k-1}).$$

Here the $\dot{a}_i$'s are $\mathbb{P}_\beta$-names for reals, and the base associated with the tag is, by definition,

$$L[T_2, \dot{a}_0, \ldots, \dot{a}_{k-1}].$$

Such a regularity tag is used only when it has a canonical admissibility witness in the sense of Definition 3.6; equivalently, the finitely many real parameters generating the base are read on a countable admissible support. In the last case, $\dot{m}$ is a name for a natural number coding the relevant set A_m, $\dot{x}$ is the section parameter, and $\dot{z}$ is the bookkeeping guess for a possible value in the x-section. The name $\dot{z}$ is disposable: at successor allowability stages it is used only when it is not selected as the protected tentative value.

The successor stages use the same hybrid product/iteration form as above. At every successor stage one first adds a fresh, independent copy of the M_1-Cohen coding-area forcing. The bookkeeping value then determines the second-step forcing over the product with this fresh reservoir coordinate. If the stage explicitly codes a real $\dot{u}$, this second step is the Jensen–Solovay almost-disjoint coding determined by $\dot{u}$. Over the current intermediate model the coding quotient is canonically isomorphic to the two-step forcing

$$\mathbb{C}_{M_1} * \dot{\mathbb{A}}_D(\dot{Y}_{\dot{g},\dot{u}}),$$

but in the global hybrid presentation the first factor is inserted as an actual product factor:

$$(\text{current forcing} \times \mathbb{C}_\beta) * \dot{\mathbb{A}}_D(\dot{Y}_{\dot{g}_\beta,\dot{u}}).$$

Here $\mathbb{C}_\beta$ is the fresh reservoir coordinate for the stage and $\dot{g}_\beta$ is its canonical generic. At relative Random, relative amoeba, ordinary Cohen, and trivial stages the same reservoir coordinate is still present, but no almost-disjoint code is written from it. The coding areas obtained from distinct reservoir coordinates are mutually almost disjoint by the standard fusion argument for $\mathbb{C}_{M_1}$.

Definition 3.8 (0-allowable local hybrid presentations)**.** *Let $F \in M_1$ be a bookkeeping function of length $\delta \leqslant \omega_2$. A sequence*

$$\mathcal{R} = \langle \mathcal{R}_\beta \mid \beta \leqslant \delta \rangle$$

over W is a 0-allowable local hybrid presentation relative to F *if it is obtained by the following recursion.*

First, $\mathcal{R}_0$ is the trivial post-branch presentation. Thus the associated quotient forcing over W is

$$\mathbb{P}_0 = \mathbf{1},$$

whereas the corresponding full forcing over M_1 is simply Br. At limit stages, $\mathcal{R}_\eta$ is the mixed-support limit of the preceding hybrid construction, and $\mathbb{P}_\eta$ is the associated post-branch forcing. Suppose that $\mathcal{R}_\beta$ has been constructed, and write $\mathbb{P}_\beta$ for its associated post-branch forcing. Choose a fresh reservoir coordinate $\mathbb{C}_\beta$, let $\dot{g}_\beta$ be its canonical generic, and put

$$\widehat{\mathbb{P}}_{\beta+1} = \mathbb{P}_\beta \times \mathbb{C}_\beta.$$

The second-step forcing $\dot{\mathbb{Q}}_\beta$ over $\widehat{\mathbb{P}}_{\beta+1}$ is determined by the decoded value of $F(\beta)$.

1. *If $F(\beta)$ is one of the relative regularity tags*

   $$(\mathrm{Rand}; \dot{a}^\beta_0, \ldots, \dot{a}^\beta_{k_\beta-1}), \qquad (\mathrm{Am}_{\mathcal{N}}; \dot{a}^\beta_0, \ldots, \dot{a}^\beta_{k_\beta-1}), \qquad (\mathrm{Am}_{\mathcal{M}}; \dot{a}^\beta_0, \ldots, \dot{a}^\beta_{k_\beta-1}),$$

   and this stage has a canonical admissibility witness in the sense of Definition 3.6, let

   $$(A^{\mathrm{reg}}_\beta, \dot{W}^{\mathrm{reg}}_\beta, B^{0,\mathrm{reg}}_\beta, \dot{B}^{\mathrm{reg}}_\beta)$$

   be that witness. Thus, for the $<_{M_1}$-least tuple of $\mathbb{P}_{A^{\mathrm{reg}}_\beta}$-names for reals $\langle \dot{a}^{\beta,\mathrm{reg}}_i \mid i < k_\beta \rangle$ reading the displayed real names, we have

   $$\dot{W}^{\mathrm{reg}}_\beta = L[T_2, \dot{a}^{\beta,\mathrm{reg}}_0, \ldots, \dot{a}^{\beta,\mathrm{reg}}_{k_\beta-1}].$$

   Let $\dot{W}^{\mathrm{reg},+}_\beta$ be the canonical $\widehat{\mathbb{P}}_{\beta+1}$-name obtained by lifting this base, equivalently the name for

   $$L[T_2, (\dot{a}^{\beta,\mathrm{reg}}_0)^\uparrow, \ldots, (\dot{a}^{\beta,\mathrm{reg}}_{k_\beta-1})^\uparrow]$$

over the stage $\widehat{\mathbb{P}}_{\beta+1}$. The second-step forcing $\dot{\mathbb{Q}}_\beta$ is the member of the following list corresponding to the tag:

$$\dot{\mathbb{B}}_{\mathrm{rand}}^{\dot W_\beta^{\mathrm{reg},+}}, \qquad \dot{\mathbb{A}}_{\mathcal{N}}^{\dot W_\beta^{\mathrm{reg},+}}, \qquad \dot{\mathbb{A}}_{\mathcal{M}}^{\dot W_\beta^{\mathrm{reg},+}}.$$

Thus the relative Random or amoeba forcing is computed in the lifted base $L[T_2, (\dot a_i^{\beta,\mathrm{reg}})^{\uparrow} : i < k_\beta]$. It is not recomputed in the larger ambient extension. The local support A_β^{reg} and the footprint name $\dot B_\beta^{\mathrm{reg}}$ are recorded as data of this stage. If the displayed regularity tag has no admissibility witness, put $\dot{\mathbb{Q}}_\beta = \mathbf{1}$.

2. *If $F(\beta)$ is an ordinary Cohen tag, then*

$$\dot{\mathbb{Q}}_\beta = \dot{\mathbb{C}}_\omega.$$

3. *If $F(\beta) = \dot w$, where $\dot w$ is a $\mathbb{P}_\beta$-name for a real, let*

$$\dot u_\beta = \dot w$$

and put

$$\dot{\mathbb{Q}}_\beta = \dot{\mathbb{A}}_D(\dot Y_{\dot g_\beta, \dot u_\beta}).$$

4. *If*

$$F(\beta) = (\dot x, \dot z, \dot m),$$

where $\dot x, \dot z$ are $\mathbb{P}_\beta$-names for reals and $\dot m$ is a $\mathbb{P}_\beta$-name for a natural number, let

$$\dot u_\beta = \langle \dot x, \dot z, \dot m \rangle$$

and put

$$\dot{\mathbb{Q}}_\beta = \dot{\mathbb{A}}_D(\dot Y_{\dot g_\beta, \dot u_\beta}).$$

5. *If*

$$F(\beta) = (\dot z_0, \dot z_1),$$

where $\dot z_0, \dot z_1$ are $\mathbb{P}_\beta$-names for reals, let (A_i, σ_i) be the canonical localized presentation of $\dot z_i$, for $i < 2$. If

$$(A_0, \sigma_0) <_{M_1} (A_1, \sigma_1),$$

let

$$\dot u_\beta = \langle \dot z_0, \dot z_1 \rangle$$

and put

$$\dot{\mathbb{Q}}_\beta = \dot{\mathbb{A}}_D(\dot Y_{\dot g_\beta, \dot u_\beta}).$$

If instead

$$(A_1, \sigma_1) <_{M_1} (A_0, \sigma_0),$$

let

$$\dot{u}_\beta = \langle \dot{z}_1, \dot{z}_0 \rangle$$

and put

$$\dot{\mathbb{Q}}_\beta = \dot{\mathbb{A}}_D(\dot{Y}_{\dot{g}_\beta, \dot{u}_\beta}).$$

If the two canonical localized presentations are equal, put

$$\dot{\mathbb{Q}}_\beta = \mathbf{1}.$$

If none of the preceding cases applies, put

$$\dot{\mathbb{Q}}_\beta = \mathbf{1}.$$

Finally define

$$\mathbb{P}_{\beta+1} = \widehat{\mathbb{P}}_{\beta+1} * \dot{\mathbb{Q}}_\beta.$$

In cases (3), (4), and the nontrivial subcases of (5), we say that stage β explicitly codes *the name* $\dot{u}_\beta$*. A forcing admitting such a presentation is called* locally allowable, *or* 0-allowable.

Lemma 3.9. *Let* $\mathcal{R}_\delta$ *be a* 0*-allowable local hybrid presentation over* W*. Then the following hold.*

1. ***Normal form.*** *Over the branch extension* W*, the post-branch quotient forcing described by* $\mathcal{R}_\delta$ *can be rearranged into an equivalent presentation of the form*
$$\mathbb{C}_{\text{res}} * \dot{\mathbb{S}}.$$
Here $\mathbb{C}_{\text{res}}$ *is the countable-support product of all fresh* M_1*-Cohen reservoir coordinates introduced at successor stages of the hybrid presentation, including stages whose second-step forcing is Random, amoeba, ordinary Cohen, or trivial. The forcing* $\dot{\mathbb{S}}$ *is a finite-support iteration of c.c.c. real-adding forcings: ordinary Cohen forcing, fixed relative Random algebras, fixed relative amoeba forcings over their associated* $L[T_2, \vec{a}]$*-bases, Jensen–Solovay almost-disjoint coding forcings computed from the relevant reservoir generics and localized names, and trivial factors. Equivalently, the full forcing* $Br \ltimes_{\text{hyb}} \mathcal{R}_\delta$ *over* M_1 *is equivalent to*
$$Br * (\mathbb{C}_{\text{res}} * \dot{\mathbb{S}}).$$

2. *The post-branch quotient* $\mathbb{P}_\delta$ *is proper and preserves* ω_1*; equivalently, the full forcing* $Br \ltimes_{\text{hyb}} \mathcal{R}_\delta$ *preserves* ω_1 *over* M_1*. Moreover, if* $\delta \leqslant \omega_2$*, then the corresponding forcing preserves* ω_2*.*

3. *Coding areas are almost disjoint: if* g *and* h *are distinct* M_1*-Cohen subsets of* ω_1*, then the corresponding coding areas* C_g *and* C_h *have bounded, hence countable, intersection. In particular this holds for the reservoir generics added at distinct successor stages of the hybrid presentation.*

4. ***Non-interference of coding blocks.*** *A coding block attached to a reservoir coordinate is read only from that reservoir area and from the corresponding almost-disjoint coding real. Later stages use fresh reservoir coordinates. Hence later forcing cannot alter the decoding of an earlier block, and a decoded area which agrees with the reservoir area of an earlier explicit coding stage yields the real intentionally written at that stage. Reservoir coordinates whose second-step forcing is Random, amoeba, ordinary Cohen, or trivial write no almost-disjoint code and therefore create no coding block.*

5. *If a stage explicitly codes a name* $\dot{u}_\beta$ *and* $u_\beta = (\dot{u}_\beta)^{G_\beta}$*, then in the next intermediate extension there is a real* r_β *such that* $\Psi(r_\beta, u_\beta)$ *holds. In particular* $\Phi(u_\beta)$ *holds at that stage. This instance of* $\Phi(u_\beta)$ *persists to all later locally allowable extensions which preserve* ω_1 *and use fresh reservoir coordinates.*

Proof. For (1), observe that the first component of every successor stage is a fresh copy of the M_1-computed ω_1-Cohen reservoir forcing. These coordinates are independent of one another and are ordered with countable support. They may therefore be moved to a single product part $\mathbb{C}_{\mathrm{res}}$. The second-step forcing at a stage is then read over the product extension. If the stage is a relative regularity stage, the forcing remains the fixed forcing computed in the lifted admissible base; it is not recomputed in the larger universe. If the stage is an explicit coding stage, the almost-disjoint forcing is computed from the lifted localized name and from the corresponding reservoir generic. Ordinary Cohen and trivial stages are unaffected. Canonical lifting of localized presentations preserves the well-order comparison clauses. This gives the stated normal form, and the forcing over M_1 is obtained by placing the initial branch product Br in front.

For (2), use the normal form from (1). The branch product Br is c.c.c. and has size $\aleph_1$. The reservoir part $\mathbb{C}_{\mathrm{res}}$ is the M_1-computed countable-support product of copies of $\mathbb{C}_{M_1}$. We use its closure before the branch product is performed: over M_1 this reservoir block is σ-closed, hence it preserves ω_1 and preserves the finite-product Suslinity of the fixed sequence $\vec{S}$. Thus, in the reservoir extension, Br is still c.c.c. After the branch product, the real-adding part $\dot{\mathbb{S}}$ is a finite-support iteration of c.c.c. forcings: ordinary Cohen forcing, fixed relative Random algebras, fixed relative amoeba forcings over admissible bases, Jensen–Solovay almost-disjoint coding forcings, and trivial factors. Hence $\dot{\mathbb{S}}$ is c.c.c. and therefore proper. This commuted presentation proves preservation for the full forcing over M_1, and therefore for the post-branch quotient over W. Notice that we do not use the post-branch reservoir quotient as a literally σ-closed forcing over W.

If $\delta \leqslant \omega_2$, the same normal form gives preservation of ω_2. Since M_1 satisfies GCH and Br has size $\aleph_1$, the branch extension W satisfies CH and $(\omega_2)^\omega = \omega_2$. The usual Δ-system argument therefore shows that the countable-support product of at most ω_2 many $\aleph_1$-sized reservoir coordinates has the $\aleph_2$-chain condition. In the reservoir extension, every real-adding iterand in $\dot{\mathbb{S}}$ has size at most $\aleph_1$: this is clear for ordinary Cohen forcing, Jensen–Solovay almost-disjoint coding, and trivial forcing, and for relative relative regularity forcing it follows because admissible bases have only $\aleph_1$-many relevant Borel codes. A second Δ-system argument therefore shows that the finite-support real-adding iteration has the $\aleph_2$-chain condition. Thus ω_2 is preserved.

Clause (3) follows from the definition of the coding areas. If $g \neq h$, let $\xi < \omega_1$ be the first coordinate at which they differ. Then no proper initial segment of g of length above ξ can equal a proper initial segment of h of length above ξ. Since ρ is one-to-one, $C_g \cap C_h$ is contained in the codes of the initial segments of length at most ξ, and is therefore bounded in ω_1.

For (4), the decoding predicate is local to a single reservoir area and to the almost-disjoint real added for the reshaped set associated with that area. Distinct reservoir areas are almost disjoint by (3), and all later stages use fresh reservoir coordinates. Thus later coding blocks may add new codes, but they do not change the old almost-disjoint pattern and cannot make a disjoint block decode the same old object. If a purported decoding uses the same reservoir area as an earlier explicit coding stage, then the David decoding on that area is the one produced at that stage, so it yields the intentionally coded real. If the reservoir area belongs to a non-coding stage, there is no corresponding almost-disjoint coding real and hence no coding block to decode.

For (5), after g_β is added, the set Y_{g_β,u_β} is exactly the David reshaping of the model code associated with u_β, and the real r_β added by $\mathbb{A}_D(Y_{g_\beta,u_\beta})$ almost-disjointly decodes this Y over every relevant countable model. Hence $\Psi(r_\beta, u_\beta)$ holds by the definition of Ψ. The persistence follows because later locally allowable stages preserve ω_1, use fresh reservoir coordinates, and do not change the already added real r_β or the initial segment of the almost-disjoint family used in the decoding. □

We next fix the branch-footprint notation used in the omission arguments. This notation is not part of the definition of admissibility.

Let $A \subseteq \delta$ be a countable admissible support, let $\vec{\sigma}$ be a finite tuple of $\mathbb{P}_A$-names for reals, and let $G_\delta \subseteq \mathbb{P}_\delta$ be generic. The names in $\vec{\sigma}$, the bookkeeping entries used on A, and the names appearing in the stages of $\mathcal{R}_A$ may use parameters from $W = M_1[H]$. Each such parameter is represented by its $<_{M_1}$-least Br-name. Let

$$B_0(A, \vec{\sigma}) \subseteq \omega_1$$

be the union of the supports of these Br-names. If $\eta \in A$ is a relative Random, measure-amoeba, or category-amoeba stage, and

$$(A^{\mathrm{reg}}_\eta, \dot{W}^{\mathrm{reg}}_\eta, B^{0,\mathrm{reg}}_\eta, \dot{B}^{\mathrm{reg}}_\eta)$$

is the canonical admissibility witness for its base, then we also put $B_\eta^{0,\mathrm{reg}}$ into $B_0(A,\vec{\sigma})$. Thus $B_0(A,\vec{\sigma})$ is countable.

Let $R(A)$ be the set of relative Random and relative amoeba stages in A, and let $E(A)$ be the set of explicit coding stages in A. If $\eta \in R(A)$, set

$$B_\eta^{\mathrm{reg},G} = (\dot{B}_\eta^{\mathrm{reg}})^{G_{A_\eta^{\mathrm{reg}}}}.$$

If $\eta \in E(A)$, let C_η^G be the coding area determined by the reservoir generic at η, and let u_η^G be the real coded at η. Define

$$\begin{aligned} U(A,\vec{\sigma},G_\delta) = B_0(A,\vec{\sigma}) \cup & \bigcup_{\eta \in R(A)} B_\eta^{\mathrm{reg},G} \\ \cup & \bigcup_{\eta \in E(A)} \mathrm{Sel}(C_\eta^G, u_\eta^G). \end{aligned}$$

Thus $U(A,\vec{\sigma},G_\delta)$, informally the branch footprint of A and $\vec{\sigma}$ in the generic extension, records the branch coordinates used by the restricted presentation and the displayed names. It is generated by countably many coding areas and a countable set of branch-product coordinates. It may have size ω_1. At a coding stage only the selected coordinates $\mathrm{Sel}(C_\eta^G, u_\eta^G)$ are included; the other member of $\{\omega\cdot\gamma+2n, \omega\cdot\gamma+2n+1\}$ is not included unless it is selected by another coding stage or belongs to a regularity-base footprint.

Lemma 3.10 (Countable localization and branch footprints)**.** *Let $\mathcal{R}_\delta$ be a 0-allowable local hybrid presentation over W, and let $\mathbb{P}_\delta$ be its associated post-branch forcing. If $\vec{\tau}$ is a finite tuple of $\mathbb{P}_\delta$-names for reals or natural numbers, then there are a countable admissible support $A \subseteq \delta$ and a tuple $\vec{\sigma}$ of $\mathbb{P}_A$-names such that*

$$\mathbb{P}_\delta \Vdash \vec{\tau} = \vec{\sigma}^{\uparrow\delta}.$$

Consequently, if $G_\delta \subseteq \mathbb{P}_\delta$ is generic and $x \in W[G_\delta] \cap 2^\omega$, then

$$x \in W[G_A]$$

for some countable admissible $A \subseteq \delta$, where $G_A = G_\delta \cap \mathbb{P}_A$.

Moreover, for every such A, every finite tuple $\vec{\sigma}$, and every G_δ, the branch footprint has the form

$$U(A,\vec{\sigma},G_\delta) = B \cup \bigcup_{j\in J} \mathrm{Sel}(C_j,u_j),$$

where J is countable, $B \in [\omega_1]^\omega$, each C_j is a reservoir coding area occurring in the local presentation, and each $u_j \in 2^\omega$. Thus the conclusions of Lemma 3.2 apply to $U(A,\vec{\sigma},G_\delta)$.

Proof. We prove the localization statement by induction on δ. We use the normal form from Lemma 3.9, but in the commuted full presentation over M_1. Thus the full forcing is equivalent to

$$\mathbb{C}_{\text{res}} * (Br * \dot{\mathbb{S}}),$$

where $\mathbb{C}_{\text{res}}$ is the M_1-computed countable-support product of the reservoir coordinates, and, in the $\mathbb{C}_{\text{res}}$-extension, $Br * \dot{\mathbb{S}}$ is a finite-support c.c.c. iteration of the branch product together with the real-adding second-step coordinates. The reservoir block is used here before the branch product. It is σ-closed over M_1, and hence adds no reals, and no countable sequences of M_1-ordinals. We do not use the post-branch reservoir quotient over $W = M_1[H]$ as a closed forcing.

Replace $\vec{\tau}$ by equivalent names in this commuted presentation. In the $\mathbb{C}_{\text{res}}$-extension, take nice names for these reals and natural numbers with respect to the c.c.c. forcing $Br * \dot{\mathbb{S}}$. Thus, for each bit of each real name, and for each natural-number name, there is a countable maximal antichain deciding the relevant value. Each condition in such an antichain has finite branch support and finite support in the real-adding second-step coordinates. Since $\mathbb{C}_{\text{res}}$ adds no countable sequences of ground ordinals, the union of the second-step supports appearing in all these antichains is represented by a countable set of hybrid coordinates. Let this set be $A_0 \subseteq \delta$. The branch coordinates which occur in the W-parameters of these names are recorded separately in B_0; they are not coordinates of the post-branch forcing.

We now close A_0 under the dependencies needed to make the restricted hybrid presentation meaningful. Given A_n, define A_{n+1} as follows. If $\eta \in A_n$ is a relative Random, measure-amoeba, or category-amoeba stage, let

$$(A^{\text{reg}}_\eta, \dot{W}^{\text{reg}}_\eta, B^{0,\text{reg}}_\eta, \dot{B}^{\text{reg}}_\eta)$$

be the canonical support witness for the regularity base at η. Thus

$$\dot{W}^{\text{reg}}_\eta = L[T_2, \dot{a}^\eta_0, \ldots, \dot{a}^\eta_{k_\eta - 1}]$$

for finitely many names for reals read on A^{reg}_η. Put $A^{\text{reg}}_\eta \subseteq A_{n+1}$. If $\eta \in A_n$ is an explicit coding stage, apply the induction hypothesis below η to the name which is coded at η, and add the resulting countable admissible support to A_{n+1}. Ordinary Cohen and trivial stages require no additional support. Finally let

$$A = \bigcup_{n<\omega} A_n.$$

Then A is countable.

By construction, every nontrivial stage kept in A has all earlier data needed to interpret its tag: regularity stages have their finite real-parameter base read on the included support, and coding stages have the coded real read on the included support. Hence A is admissible in the sense of Definition 3.4, and the canonical embedding identifies $\mathbb{P}_A$ with a complete subforcing of $\mathbb{P}_\delta$. The nice names chosen above mention only stages in A, after replacing every kept stage by its restricted version. Therefore they define a tuple $\vec{\sigma}$ of $\mathbb{P}_A$-names such that

$$\mathbb{P}_\delta \Vdash \vec{\tau} = \vec{\sigma}^{\uparrow\delta}.$$

This proves the localization statement. The assertion for a real $x \in W[G_\delta]$ follows by applying the first part to a name for x.

It remains to record the footprint form. By definition, $B_0(A, \vec{\sigma})$ is countable. Each explicit coding stage in A contributes one set $\mathrm{Sel}(C, u)$. If $\eta \in A$ is a relative Random, measure-amoeba, or category-amoeba stage, then the canonical support witness for its base contributes

$$B_\eta^{0,\mathrm{reg}} \cup \bigcup_{\nu \in E(A_\eta^{\mathrm{reg}})} \mathrm{Sel}(C_\nu, u_\nu).$$

Here A_η^{reg} is the countable support on which the finitely many real parameters generating the displayed $L[T_2, \vec{a}]$-base are read, and $B_\eta^{0,\mathrm{reg}}$ is the countable branch-product support of the W-parameters occurring in those names. Since A is countable, all selected coding-area contributions can be enumerated as

$$(C_j, u_j)_{j \in J} \qquad \text{with } J \text{ countable},$$

together with one countable set $B \subseteq \omega_1$. Hence

$$U(A, \vec{\sigma}, G_\delta) = B \cup \bigcup_{j \in J} \mathrm{Sel}(C_j, u_j).$$

Lemma 3.2 applies to every set of this form. □

Lemma 3.11. *Let $\mathcal{R}_\delta$ be a 0-allowable local hybrid presentation over W, let $G_\delta \subseteq \mathbb{P}_\delta$ be generic, and let $A \subseteq \delta$ be a countable admissible support. Let $\vec{\sigma}$ be a finite tuple of $\mathbb{P}_A$-names for reals. Suppose that $\xi < \omega_1$ satisfies*

$$\xi \notin U(A, \vec{\sigma}, G_\delta).$$

Let $H \subseteq Br$ be the branch-product generic used to form W, and set

$$W_\xi = M_1[H \upharpoonright (\omega_1 \backslash \{\xi\})].$$

Then $\mathcal{R}_A$ is interpretable over W_ξ. If $G_A = G_\delta \cap \mathbb{P}_A$, then G_A is generic over W_ξ, the interpretations of the names in $\vec{\sigma}$ belong to $W_\xi[G_A]$, and

$$W_\xi[G_A] \models \text{“}S_\xi \textit{ is Suslin.”}$$

Proof. We argue by induction over the restricted presentation $\mathcal{R}_A$. Since $\xi \notin B_0(A, \vec{\sigma})$, every ground-model parameter from $W = M_1[H]$ used by $\vec{\sigma}$, by the bookkeeping on A, or by a stage of $\mathcal{R}_A$, is represented by a $<_{M_1}$-least Br-name whose support omits ξ. Hence all such parameters are in W_ξ. Moreover the product on the branch coordinates different from ξ preserves S_ξ, by independence of $\vec{S}$.

Suppose first that $\eta \in A$ is a relative Random, measure-amoeba, or category-amoeba stage. Let

$$(A_\eta^{\mathrm{reg}}, \dot{W}_\eta^{\mathrm{reg}}, B_\eta^{0,\mathrm{reg}}, \dot{B}_\eta^{\mathrm{reg}})$$

be the admissibility witness for its base. Since A is admissible,

$$A_\eta^{\mathrm{reg}} \subseteq A \cap \eta.$$

Also

$$B_\eta^{0,\mathrm{reg}} \subseteq B_0(A, \vec{\sigma}) \quad \text{and} \quad (\dot{B}_\eta^{\mathrm{reg}})^{G_{A_\eta^{\mathrm{reg}}}} \subseteq U(A, \vec{\sigma}, G_\delta).$$

Thus ξ is outside the branch footprint of the base. By Definition 3.6, this witness is obtained from finitely many $\mathbb{P}_{A_\eta^{\mathrm{reg}}}$-names for reals

$$\dot{a}_0^\eta, \ldots, \dot{a}_{k_\eta - 1}^\eta$$

and

$$\dot{W}_\eta^{\mathrm{reg}} = L[T_2, \dot{a}_0^\eta, \ldots, \dot{a}_{k_\eta-1}^\eta].$$

The set $B_\eta^{0,\mathrm{reg}}$ records the branch-product supports of the W-parameters occurring in the codes of these real names, and $(\dot{B}_\eta^{\mathrm{reg}})^{G_{A_\eta^{\mathrm{reg}}}}$ records the branch coordinates generated by the explicit coding stages in A_η^{reg}. Since both are contained in $U(A, \vec{\sigma}, G_\delta)$ and $\xi \notin U(A, \vec{\sigma}, G_\delta)$, the finite tuple

$$\langle \dot{a}_i^\eta \mid i < k_\eta \rangle$$

is interpreted in $W_\xi[G_{A_\eta^{\mathrm{reg}}}]$ with the same value as in $W[G_{A_\eta^{\mathrm{reg}}}]$. Hence the base interpreted in the ξ-omitting model is the same model

$$L[T_2, a_0^\eta, \ldots, a_{k_\eta-1}^\eta]$$

as in the full extension; here $T_2 \in M_1$ is fixed ground-model data. The iterand at η is therefore the same relative Random, measure-amoeba, or category-amoeba forcing computed in this base; it is not recomputed in the ambient model. The relative Random algebra is Knaster, and the two amoeba forcings used here are σ-linked. Hence the iterand preserves S_ξ.

If $\eta \in A$ is an ordinary Cohen or trivial stage, the conclusion is immediate. The reservoir coordinate at η is the fixed M_1-Cohen forcing and is independent of the branch coordinate ξ; in the normal form of Lemma 3.9 it is placed before the branch product and preserves S_ξ.

It remains to consider an explicit coding stage $\eta \in A$. Let $\dot{u}_\eta$ be the name coded at η, and let C^G_η be the coding area of the reservoir generic. By admissibility, $\dot{u}_\eta$ is read on an earlier support contained in A. The branch coordinates used by the David code at this stage are precisely

$$\mathrm{Sel}(C^G_\eta, u^G_\eta).$$

Since this set is contained in $U(A, \vec{\sigma}, G_\delta)$ and $\xi \notin U(A, \vec{\sigma}, G_\delta)$, the branch b_ξ is not used in forming Y_{g_η, u_η}. Hence the same reshaped set and the same almost-disjoint coding forcing are computed over the ξ-omitting model. The Jensen–Solovay almost-disjoint forcing is Knaster, and so preserves S_ξ.

At limit stages the restricted presentation uses mixed support. The support in A is countable on reservoir coordinates and finite on the c.c.c. real-adding coordinates at each condition. The preceding successor analysis therefore iterates along A. Thus $\mathcal{R}_A$ is defined over W_ξ, G_A is generic over W_ξ, the tuple $\vec{\sigma}^{G_A}$ belongs to $W_\xi[G_A]$, and no stage of the restricted forcing adds a branch through S_ξ. Hence S_ξ remains Suslin in $W_\xi[G_A]$. ☐

Lemma 3.12 (No unwanted Φ-codes)**.** *Let $\mathcal{R}_\delta$ be a 0-allowable local hybrid presentation over W, and let $G_\delta \subseteq \mathbb{P}_\delta$ be generic. Suppose that $w \in W[G_\delta] \cap 2^\omega$ and*

$$W[G_\delta] \models \Phi(w).$$

Then there is an explicit coding stage $\beta < \delta$ with coded name $\dot{u}_\beta$ such that

$$w = (\dot{u}_\beta)^{G_\beta}.$$

Equivalently, Random stages, amoeba stages, ordinary Cohen stages, trivial stages, and coding stages for other reals do not create new instances of Φ.

Proof. Fix $r \in W[G_\delta]$ such that $\Psi(r, w)$. Put

$$m_\infty = \mathcal{J}^{M_1}_{\omega_1}.$$

By Lemma 3.3, the model $L[r, m_\infty]$ decodes a coding area $C_r \subseteq \omega_1$ and a selected branch pattern. Thus $|C_r| = \omega_1$, and for every $\gamma \in C_r$ and $n < \omega$, if

$$\xi = \tau_w(\gamma, n),$$

then $L[r, m_\infty]$ contains an ω_1-branch through S_ξ.

First suppose that $r \in W$. Since $W = M_1[H]$ is obtained by the finite-support branch product, there is $B \in [\omega_1]^\omega$ such that $r \in M_1[H \restriction B]$. Since $w \in L[r, m_\infty]$, also $w \in M_1[H \restriction B]$. Choose $\gamma \in C_r$ such that

$$\{\omega \cdot \gamma + k \mid k < \omega\} \cap B = \varnothing.$$

Choose $n < \omega$, and set $\xi = \tau_w(\gamma, n)$. Then $\xi \notin B$. By independence of $\vec{S}$, the tree S_ξ is Suslin in $M_1[H \restriction B]$. But

$$L[r, m_\infty] \subseteq M_1[H \restriction B]$$

contains an ω_1-branch through S_ξ, a contradiction. Hence $r \notin W$.

Choose a countable admissible support $A \subseteq \delta$ and a $\mathbb{P}_A$-name σ such that

$$\sigma^{G_A} = r, \qquad G_A = G_\delta \cap \mathbb{P}_A.$$

Enlarge A, if necessary, so that it contains the local supports needed for this name. Let $\vec{\sigma} = (\sigma)$. By the definition of the footprint, there are a countable set J, a countable set $B \subseteq \omega_1$, coding areas $(C_j : j \in J)$, and reals $(u_j : j \in J)$, such that

$$U(A, \vec{\sigma}, G_\delta) = B \cup \bigcup_{j \in J} \mathrm{Sel}(C_j, u_j).$$

Each pair (C_j, u_j) comes from an explicit coding stage of $\mathcal{R}_\delta$. The set B contains the branch-product supports of the W-parameters in the names under consideration and the countable parts of the regularity-base witnesses. The coding areas $(C_j : j \in J)$ are pairwise almost disjoint.

If there is $j \in J$ such that $u_j = w$, then w is the real explicitly coded at the corresponding stage, and we are done. We therefore assume

$$\forall j \in J \quad u_j \neq w.$$

Let

$$J_1 = \{j \in J \mid C_r \cap C_j \text{ is unbounded in } \omega_1\}.$$

If $J_1 \neq \varnothing$, choose $j_0 \in J_1$. Then $u_{j_0} \neq w$. Choose $n < \omega$ with

$$u_{j_0}(n) \neq w(n).$$

By Lemma 3.2, applied to the countable family $(C_j : j \in J)$, the set B, the set C_r, and the index j_0, there is $\gamma \in C_r \cap C_{j_0}$ such that

$$\tau_w(\gamma, n) \notin U(A, \vec{\sigma}, G_\delta).$$

Set $\xi = \tau_w(\gamma, n)$.

If $J_1 = \varnothing$, then every $C_r \cap C_j$, $j \in J$, is bounded. By Lemma 3.2 there is $\gamma \in C_r$ such that

$$\{\omega \cdot \gamma + k \mid k < \omega\} \cap U(A, \vec{\sigma}, G_\delta) = \varnothing.$$

Choose $n < \omega$, and set $\xi = \tau_w(\gamma, n)$.

In both cases

$$\xi \notin U(A, \vec{\sigma}, G_\delta),$$

and $L[r, m_\infty]$ contains an ω_1-branch through S_ξ. By Lemma 3.11, the restricted extension $W_\xi[G_A]$ contains r, and S_ξ remains Suslin in $W_\xi[G_A]$. Since $m_\infty \in M_1 \subseteq W_\xi[G_A]$, we have

$$L[r, m_\infty] \subseteq W_\xi[G_A].$$

This contradicts the Suslinity of S_ξ in $W_\xi[G_A]$. The contradiction shows that the earlier case must occur: w is explicitly coded at some stage $\beta < \delta$. $\square$

We now define the derivative hierarchy of allowable forcings. The point of the hierarchy is to separate genuine candidates for a uniformizing value from guesses which can be destroyed by further allowable forcing. A successor step asks the following local question. Given the bookkeeping data (x, A_m) in the current extension, is there already, in the local subextension determined by the canonical support of these data, a real y such that (x, y) is in the x-section of A_m and such that lower-rank allowable tails cannot force it out of that section? If the answer is yes, y becomes a tentative value of the least possible rank, and the construction codes some other real on the same section. If the answer is no, the construction treats the bookkeeping value as a guess and codes that guess. The hierarchy is defined by transfinite recursion. Limit levels are pure intersections; successor levels are obtained by the local test just described.

For ρ-allowability we work with pairs

$$(\mathcal{R}, \dot{I}) \in W,$$

where $\mathcal{R}$ is a mixed-support local hybrid presentation over W and $\dot{I}$ is a name, over the terminal post-branch forcing associated with $\mathcal{R}$, for a set of quadruples (x, y, m, ξ). After interpretation by a generic filter, I is the set of tentative uniformizing values, and the ordinal ξ records the rank at which the value was selected. The base level is

$$0\text{-allowable} = \text{locally allowable},$$

with $\dot{I}$ the canonical name for the empty set.

Definition 3.13 (Limit λ-allowability)**.** *Assume that λ is a nonzero limit ordinal and that ζ-allowability has already been defined for every $\zeta < \lambda$. Let $\mathcal{A}_\zeta^W$ denote the class of pairs $(\mathcal{R}, \dot{I}) \in W$ which are ζ-allowable over W. We define*

$$\mathcal{A}_\lambda^W = \bigcap_{\zeta<\lambda} \mathcal{A}_\zeta^W.$$

Thus $(\mathcal{R}, \dot{I})$ is λ-allowable *iff it is ζ-allowable for every $\zeta < \lambda$.*

Definition 3.14 (Relative ρ-allowable extensions)**.** *Assume that ρ-allowability has been defined. If $(\mathcal{R}, \dot{I})$ and $(\mathcal{S}, \dot{J})$ are pairs belonging to W, we write*

$$(\mathcal{S}, \dot{J}) \rhd_\rho (\mathcal{R}, \dot{I})$$

if $\mathcal{S}$ is obtained by appending to $\mathcal{R}$ a mixed-support local hybrid tail which satisfies the same ρ-allowability clauses, starting with the auxiliary name $\dot{I}$ instead of the empty name. Let P and P' be the terminal post-branch forcings associated with $\mathcal{R}$ and $\mathcal{S}$, respectively. If $G \subseteq P$ is generic, then P'/G is called a ρ-allowable tail *over the interpreted pair $(\mathcal{R}, \dot{I})^G$.*

Definition 3.15 (Successor ρ-allowability)**.** *Assume that $\rho = \alpha + 1$ and that ζ-allowability and relative ζ-allowability have been defined for every $\zeta < \rho$. Let $F \in M_1$ be a bookkeeping function of length $\delta \leqslant \omega_2$. We say that $(\mathcal{R}_\delta, \dot{I}_\delta)$ is* ρ-allowable relative to *F if it is obtained by the following recursion in W.*

At each stage $\beta \leqslant \delta$ we construct a hybrid presentation $\mathcal{R}_\beta$ and a $\mathbb{P}_\beta$-name $\dot{I}_\beta$, where $\mathbb{P}_\beta$ denotes the post-branch forcing associated with $\mathcal{R}_\beta$. We start with W and with $\dot{I}_0$ the canonical name for the empty set. At a limit stage $\eta \leqslant \delta$, $\mathcal{R}_\eta$ is the mixed-support limit of the preceding hybrid presentations, and $\dot{I}_\eta$ is the canonical $\mathbb{P}_\eta$-name forced to be the union of the earlier interpreted auxiliary names.

Suppose that $\beta < \delta$ and that $(\mathcal{R}_\beta, \dot{I}_\beta)$ has already been constructed. The value $F(\beta)$ is decoded relative to $\mathbb{P}_\beta$. If the decoded value is a relative Random tag, a relative amoeba tag, a Cohen tag, a direct coding tag $\dot{w}$, or a well-order tag $(\dot{z}_0, \dot{z}_1)$, then the next stage is the hybrid operation prescribed in Definition 3.8, and $\dot{I}_{\beta+1}$ is the canonical lift of $\dot{I}_\beta$.

It remains to describe the uniformization tag. Suppose that

$$F(\beta) = (\dot{x}, \dot{z}, \dot{m}),$$

where $\dot{x}, \dot{z}$ are $\mathbb{P}_\beta$-names for reals and $\dot{m}$ is a $\mathbb{P}_\beta$-name for a natural number. Let

$$A = \mathrm{supp}_{\mathrm{loc}}((\dot{x}, \dot{m}), \mathbb{P}_\beta).$$

A candidate value in the test below is a $\mathbb{P}_A$-name $\dot{y}_0$ for a real, lifted canonically to a $\mathbb{P}_\beta$-name. We distinguish two alternatives.

(a) *There are an ordinal $\zeta < \rho$ and a $\mathbb{P}_A$-name $\dot{y}_0$ for a real such that*

$$\mathbb{P}_\beta \Vdash (\dot{x}, \dot{y}_0^{\uparrow\beta}) \in A_{\dot{m}},$$

and no relative ζ-allowable extension in W can force this pair out of the relevant section. More explicitly, for every relative ζ-allowable extension

$$(\mathcal{S}, \dot{J}) \trianglerighteq_\zeta (\mathcal{R}_\beta, \dot{I}_\beta)$$

in W, write $P_{\mathcal{S}}$ for the terminal post-branch forcing associated with $\mathcal{S}$. If the displayed names are lifted to $P_{\mathcal{S}}$, then

$$\mathbb{P}_\beta \Vdash \text{“there is no condition in } P_{\mathcal{S}}/\dot{G}_\beta \text{ which forces } (\dot{x}^{\uparrow\mathcal{S}}, \dot{y}_0^{\uparrow\mathcal{S}}) \notin A_{\dot{m}^{\uparrow\mathcal{S}}}.\text{”}$$

If such witnesses exist, choose first the least possible ζ, and then choose the $<_{M_1}$-least localized presentation of a witnessing name $\dot{y}_0$.

In this case $\dot{y}_0$ is declared to be a tentative value of rank ζ. The next forcing does not code this tentative value. Instead it codes another real on the same section: if the bookkeeping guess $\dot{z}$ is forced to be different from $\dot{y}_0^{\uparrow\beta}$, use $\dot{z}$; otherwise use the $<_{M_1}$-least localized name $\dot{z}'$ for a *real forced to be different from* $\dot{y}_0^{\uparrow\beta}$. *Denote the resulting name by* $\dot{u}$. *The next coding operation is*

$$\mathrm{Code}(\langle \dot{x}, \dot{u}, \dot{m} \rangle),$$

understood in the hybrid sense of Definition 3.8. The auxiliary name is updated by letting $\dot{I}_{\beta+1}$ *be the canonical* $\mathbb{P}_{\beta+1}$*-name forced to be*

$$\dot{I}_\beta^{\dot{G}_\beta} \cup \{(\dot{x}^{\dot{G}_\beta}, (\dot{y}_0^{\uparrow\beta})^{\dot{G}_\beta}, \dot{m}^{\dot{G}_\beta}, \check{\zeta})\}.$$

(b) *If clause (a) fails, then the construction guesses. The bookkeeping guess* $\dot{z}$ *is used, unless the bookkeeping code does not provide a usable real name, in which case we use the* $<_{M_1}$*-least default real name. The next coding operation is*

$$\mathrm{Code}(\langle \dot{x}, \dot{z}, \dot{m} \rangle),$$

again understood as the corresponding hybrid product/iteration step, and no new tentative value is added:

$$\dot{I}_{\beta+1} = \dot{I}_\beta$$

up to the canonical lift to the longer hybrid presentation.

If the decoded bookkeeping value is none of the recognized tags, the stage is trivial and $\dot{I}_{\beta+1}$ *is the canonical lift of* $\dot{I}_\beta$. *If there is a bookkeeping function* $F \in M_1$ *such that* $(\mathcal{R}, \dot{I})$ *is* ρ*-allowable relative to* F*, then we simply say that* $(\mathcal{R}, \dot{I})$ *is* ρ*-allowable.*

Lemma 3.16 (Shrinking of allowability)**.** *Work in* W. *Let* $\alpha < \beta$. *Let* $\mathcal{R}$ *be a hybrid presentation, and let* $\mathbb{P}$ *be its terminal post-branch forcing. Suppose that* $\mathcal{R}$ *is* β*-allowable, say there is a* $\mathbb{P}$*-name* $\dot{I}^\beta$ *such that* $(\mathcal{R}, \dot{I}^\beta)$ *is* β*-allowable. Then there is a* $\mathbb{P}$*-name* $\dot{I}^\alpha$ *such that* $(\mathcal{R}, \dot{I}^\alpha)$ *is* α*-allowable. In particular, the classes of underlying allowable hybrid presentations are decreasing with the rank of allowability.*

Proof. We prove the statement for the underlying hybrid presentation. The auxiliary name is allowed to change, and in general it should change, because the set of tentative values records the ranks at which values were selected.

If β is a limit ordinal, the assertion is immediate from Definition 3.13. If α is a nonzero limit, it is enough to prove the assertion for all $\xi < \alpha$. Thus it remains to consider the case in which β is a successor ordinal and α is either 0 or a successor ordinal.

Write $\beta = \theta + 1$, and let $F \in M_1$ witness that $(\mathcal{R}_\delta, \dot{I}_\delta^\beta)$ is β-allowable. We construct a new M_1-bookkeeping function F^α such that the α-allowable recursion guided by F^α produces the same underlying hybrid presentation

$$\langle \mathcal{R}_\eta \mid \eta \leqslant \delta \rangle.$$

At trivial, relative Random, relative amoeba, Cohen, direct coding, and well-order coding stages we keep the same bookkeeping entry. These clauses are independent of the rank of allowability.

At a uniformization stage, suppose first that the original construction uses clause (b). Then no witness for clause (a) exists at any rank below β, hence none exists at any rank below α. The α-construction therefore also uses clause (b), coding the same real.

Suppose next that the original construction uses clause (a). Let $\xi < \beta$ be the least rank witnessing clause (a), let $\dot{y}_0$ be the chosen local name, and let $\dot{u}$ be the name actually coded by the stage. If $\xi < \alpha$, then the same witness is available in the α-allowable construction, and the same name $\dot{u}$ is coded. If $\alpha \leqslant \xi < \beta$, then by minimality of ξ there is no witness to clause (a) of rank below α. In this case we alter the bookkeeping entry so that it decodes to the uniformization tuple

$$(\dot{x}, \dot{u}, \dot{m}),$$

where $\dot{u}$ is the name actually coded by the original construction. If $\alpha = 0$, this is a 0-allowable coding stage. If α is a successor, clause (a) fails below α, so clause (b) prescribes the same coding operation. Thus the same underlying forcing stage is reproduced.

This defines F^{α} recursively. The definition uses only M_1-codes for the original bookkeeping entries, the canonical localized presentations, the selected names, and the relevant ordinals. Hence $F^{\alpha} \in M_1$, and the α-allowable construction guided by F^{α} produces the same underlying hybrid presentation. □

Lemma 3.17 (Allowability of tails). *Let* $(\mathcal{R}_\delta, \dot{I}_\delta)$ *be* ρ*-allowable over* W*, witnessed by a bookkeeping function* $F \in M_1$*, and let*

$$(\mathcal{R}_\gamma, \dot{I}_\gamma), \qquad \gamma \leqslant \delta,$$

be the canonical initial hybrid presentations and auxiliary names produced by this construction. If $\beta \leqslant \gamma \leqslant \delta$*, then*

$$(\mathcal{R}_\gamma, \dot{I}_\gamma) \trianglerighteq_\rho (\mathcal{R}_\beta, \dot{I}_\beta)$$

as a relative ρ*-allowable extension in* W*. Consequently, if* $G_\beta \subseteq \mathbb{P}_\beta$ *is generic over* W*, then the quotient*

$$\mathbb{P}_\gamma / G_\beta$$

is a ρ*-allowable tail over* $W[G_\beta]$*. If* $\xi < \rho$*, then the same quotient also has a* W*-presentation as a relative* ξ*-allowable tail. Finally, relative allowable extensions are transitive.*

Proof. Restrict the bookkeeping construction witnessing the ρ-allowability of $(\mathcal{R}_\delta, \dot{I}_\delta)$ to the interval $[\beta, \gamma)$. This restricted construction starts with the already built pair $(\mathcal{R}_\beta, \dot{I}_\beta)$ and then appends exactly the same hybrid steps which occur in the original construction between stages β and γ. At successor stages, the decision is made from the current full hybrid forcing, the current auxiliary name, the decoded bookkeeping entry, the localized presentations of the names involved, and the forcing relation over that current hybrid forcing. These are exactly the data available in the relative construction starting from $(\mathcal{R}_\beta, \dot{I}_\beta)$. If the stage is a relative Random or amoeba stage, its admissibility witness

$$(A_{\mathrm{rb}}, \dot{W}_{\mathrm{rb}}, B^0_{\mathrm{rb}}, \dot{B}_{\mathrm{rb}})$$

is carried with the stage: the tail uses the forcing computed in $\dot{W}_{\mathrm{rb}}$, and the lifted footprint name records the branch coordinates generated by the local support. At limit stages both constructions take the same mixed-support hybrid limit and the same canonical union of the preceding auxiliary names.

The quotient assertion follows by interpreting the relative presentation in W by a $\mathbb{P}_\beta$-generic filter. The assertion for lower ranks follows from Lemma 3.16, applied to the relative tail. Transitivity is obtained by concatenating the two relative tail presentations, renaming reservoir coordinates to fresh coordinates if necessary. □

Lemma 3.18 (Products of allowable local hybrid presentations)**.** *Let ρ be an ordinal. For $i < 2$, suppose that*

$$(\mathcal{R}^i, \dot{I}^i)$$

is ρ-allowable over W, witnessed by a bookkeeping function $F_i \in M_1$. Let $H \subseteq Br$ be the fixed branch generic used to form $W = M_1[H]$. For $i < 2$, let P^i be the terminal post-branch forcing over W represented by $\mathcal{R}^i$. Then the ordinary product $P^0 \times P^1$, computed over W, has a canonical ρ-allowable local hybrid presentation.

More precisely, there are a local hybrid presentation $\mathcal{R}^\otimes$ over W, an auxiliary name $\dot{I}^\otimes$ over its terminal post-branch forcing, and a bookkeeping function $F^\otimes \in M_1$, definable from F_0 and F_1, such that

$$(\mathcal{R}^\otimes, \dot{I}^\otimes)$$

is ρ-allowable over W, witnessed by $F^\otimes$, and its associated terminal post-branch forcing is canonically forcing equivalent to $P^0 \times P^1$.

Proof. We argue by induction on ρ. Write

$$\mathcal{R}^i = \langle \mathcal{R}^i_\beta \mid \beta \leqslant \delta_i \rangle \qquad (i < 2),$$

and let P^i_β denote the post-branch forcing over W represented by $\mathcal{R}^i_\beta$; thus $P^i = P^i_{\delta_i}$. The product presentation $\mathcal{R}^\otimes$ is obtained by first running $\mathcal{R}^0$, and then running a shifted copy of $\mathcal{R}^1$. Every name $\dot\tau$ occurring in the second block is replaced by its canonical lift $\dot\tau^{\uparrow}$ to the corresponding product initial segment, and all reservoir coordinates in the second block are renamed to fresh coordinates. Let $P^{\otimes}_{\gamma}$ denote the post-branch forcing associated with the initial product presentation $\mathcal{R}^{\otimes}_{\gamma}$. Thus, for every $\beta \leqslant \delta_1$,

$$P^{\otimes}_{\delta_0+\beta} \cong P^0 \times P^1_{\beta}.$$

The auxiliary names are defined by

$$\dot I^{\otimes}_{\delta_0+\beta} = \dot I^0 \cup (\dot I^1_{\beta})^{\uparrow}$$

with the evident interpretation at stages $\leqslant \delta_0$. Limit stages use the same mixed-support limits as the two original presentations.

For $\rho = 0$, this concatenation witnesses 0-allowability. The ordinary Cohen, direct coding, well-order coding, and trivial stages are lifted second-block stages. If a shifted second-block stage is a relative regularity stage with tag

$$(\mathrm{Rand}; \dot a_0, \ldots, \dot a_{k-1}), \quad (\mathrm{Am}_{\mathcal{N}}; \dot a_0, \ldots, \dot a_{k-1}), \quad \text{or} \quad (\mathrm{Am}_{\mathcal{M}}; \dot a_0, \ldots, \dot a_{k-1}),$$

then the product presentation uses the shifted tag obtained by lifting the finitely many real names to the product initial segment. If the original stage has canonical admissibility witness

$$(A_{\mathrm{rb}}, \dot W_{\mathrm{rb}}, B^0_{\mathrm{rb}}, \dot B_{\mathrm{rb}}),$$

where

$$\dot W_{\mathrm{rb}} = L[T_2, \dot b_0, \ldots, \dot b_{k-1}]$$

for the least tuple of names reading the displayed real parameters, then the shifted stage has the lifted witness obtained from

$$A^{\otimes}_{\mathrm{rb}}, \qquad \langle \dot b^{\uparrow}_i \mid i < k\rangle, \qquad B^0_{\mathrm{rb}}, \qquad \dot B^{\uparrow}_{\mathrm{rb}}.$$

Here $A^{\otimes}_{\mathrm{rb}}$ is the shifted copy of A_{rb} in the second block, each $\dot b^{\uparrow}_i$ is the canonical lift of $\dot b_i$ to the product initial segment, and $\dot B^{\uparrow}_{\mathrm{rb}}$ is the canonical lift of the branch-footprint name. The lifted regularity base is therefore

$$L[T_2, \dot b^{\uparrow}_0, \ldots, \dot b^{\uparrow}_{k-1}]$$

(with the evident meaning $L[T_2]$ if $k = 0$). The Random or amoeba forcing is computed in this lifted base, not in the larger product extension. Hence the displayed product equivalence is preserved at the shifted regularity stage.

Now assume $\rho = \alpha + 1$. By the successor uniformization clause we mean the two alternatives (a) and (b) in Definition 3.15, evaluated at a uniformization tag

$$(\mathrm{UF}, \dot x, \dot z, \dot m).$$

Thus clause (a) searches through ranks $\zeta < \rho$ for a localized name $\dot{y}$ which is forced into the relevant $A_{\dot{m}}$-section and cannot be forced out by any relative ζ-allowable tail; clause (b) is used when no such witness exists.

Consider a shifted uniformization stage $\delta_0 + \beta$, and let

$$(\mathrm{UF}, \dot{x}, \dot{z}, \dot{m})$$

be the corresponding second-block bookkeeping value. For $\zeta < \rho$ and a localized P^1_β-name $\dot{y}$, write

$$\mathsf{Kick}^1_\beta(\zeta, \dot{y})$$

for the assertion that there are a relative ζ-allowable extension

$$(\mathcal{S}, \dot{J}) \rhd_\zeta (\mathcal{R}^1_\beta, \dot{I}^1_\beta)$$

and, writing $P_{\mathcal{S}}$ for the terminal post-branch forcing associated with $\mathcal{S}$, a quotient condition $p \in P_{\mathcal{S}}/\dot{G}^1_\beta$ such that

$$p \Vdash (\dot{x}^{\uparrow\mathcal{S}}, \dot{y}^{\uparrow\mathcal{S}}) \notin A_{\dot{m}^{\uparrow\mathcal{S}}}.$$

Define $\mathsf{Kick}^{\otimes}_\beta(\zeta, \dot{y}^{\uparrow})$ analogously over the product initial segment

$$(\mathcal{R}^{\otimes}_{\delta_0+\beta}, \dot{I}^{\otimes}_{\delta_0+\beta}).$$

Then

$$\mathsf{Kick}^1_\beta(\zeta, \dot{y}) \quad \Longleftrightarrow \quad \mathsf{Kick}^{\otimes}_\beta(\zeta, \dot{y}^{\uparrow}).$$

The implication from left to right follows by taking the product of the fixed first block with the displayed relative ζ-allowable tail and applying the induction hypothesis. The implication from right to left follows by restricting the witnessing product tail to the complete subpresentation generated by the lifted second-block support of $(\dot{x}, \dot{y}, \dot{m})$, using Lemmas 3.17 and 3.16.

Similarly,

$$P^1_\beta \Vdash (\dot{x}, \dot{y}) \in A_{\dot{m}} \quad \Longleftrightarrow \quad P^0 \times P^1_\beta \Vdash (\dot{x}^{\uparrow}, \dot{y}^{\uparrow}) \in A_{\dot{m}^{\uparrow}}.$$

Hence the set of clause (a) witnesses in the product construction is exactly the lift of the set of clause (a) witnesses in the second construction. Thus clause (a) holds in the shifted product stage iff it holds in the original second-block stage. When it holds, the least rank and the $<_{M_1}$-least localized witness are the lifted ones. When it fails, clause (b) uses the lifted bookkeeping guess $\dot{z}^{\uparrow}$. The auxiliary name is updated by

$$\dot{I}^0 \cup (\dot{I}^1_{\beta+1})^{\uparrow}.$$

This proves the successor step.

Finally let λ be a nonzero limit ordinal. If both inputs are λ-allowable, then by the induction hypothesis the product presentation is ζ-allowable for every $\zeta < \lambda$. Since

$$\mathcal{A}^W_\lambda = \bigcap_{\zeta<\lambda} \mathcal{A}^W_\zeta,$$

the product presentation is λ-allowable. The bookkeeping witness at a limit level is obtained by packaging the lower-level witnesses together with the concatenation description. Thus $F^{\otimes} \in M_1$ is definable from F_0 and F_1. $\square$

Lemma 3.19 (Tentative values remain in their sections)**.** *Let* $(\mathcal{R}, \dot{I})$ *be* α*-allowable over* W *with respect to an* M_1*-bookkeeping function* F*, and let* $\mathbb{P}$ *be the terminal post-branch forcing associated with* $\mathcal{R}$*. Let* $G \subseteq \mathbb{P}$ *be generic over* W*, and let* $I^G = \dot{I}^G$*. If*

$$(x, y, m, \xi) \in I^G$$

for some $\xi < \alpha$*, then*

$$W[G] \models (x, y) \in A_m.$$

Moreover, for every relative ξ*-allowable extension in* W

$$(\mathcal{S}, \dot{J}) \rhd_\xi (\mathcal{R}, \dot{I}),$$

the quotient over G *forces preservation of this membership: if* P' *is the terminal post-branch forcing associated with* $\mathcal{S}$*, then*

$$P'/G \Vdash (x, y) \in A_m.$$

Proof. Suppose that β is the stage at which the tuple (x, y, m, ξ) is added to the auxiliary name. At that stage clause (a) of Definition 3.15 was used. Let

$$\dot{x}_\beta, \qquad \dot{y}_\beta, \qquad \dot{m}_\beta$$

be the canonical names chosen there, and let $\xi_\beta = \xi$. Clause (a) gives

$$\mathbb{P}_\beta \Vdash (\dot{x}_\beta, \dot{y}_\beta) \in A_{\dot{m}_\beta},$$

and it also gives the preservation statement saying that no relative ξ_β-allowable tail over the stage-β initial segment can force the lifted pair out of the relevant section.

By Lemma 3.17, the interval from stage β to the terminal forcing is a relative ξ_β-allowable tail. Therefore no condition in the terminal forcing can force

$$(\dot{x}_\beta, \dot{y}_\beta) \notin A_{\dot{m}_\beta}.$$

The forcing theorem gives the membership in the final extension. If $(\mathcal{S}, \dot{J}) \rhd_\xi (\mathcal{R}, \dot{I})$ is any further relative ξ-allowable extension, then the concatenation of the interval from β to the terminal forcing with this further extension is again a relative ξ-allowable extension over stage β. The same preservation statement therefore rules out any quotient condition forcing the pair out of A_m. $\square$

Stabilization of the allowability hierarchy. We next pass from the transfinite derivative hierarchy to its stabilized level. The following convention is used only to make the derivative classes into subsets of one fixed set. It is unrelated to the projective coding predicate Φ.

Two pairs $(\mathcal{R}, \dot{I})$ and $(\mathcal{S}, \dot{J})$, consisting of a local hybrid presentation over W and its auxiliary name, are identified if their associated forcings are canonically isomorphic over W, the isomorphism preserves the order of stages and sends the auxiliary name $\dot{I}$ to $\dot{J}$. This identification only removes harmless choices such as the names of fresh reservoir coordinates. For each equivalence class we take its $<_{M_1}$-least representative and denote it by

$$\mathrm{rep}_W(\mathcal{R}, \dot{I}).$$

Let $\mathcal{C}^W$ be the set of all such representatives for pairs of the form used in the definitions of ρ-allowability. For every ordinal ρ put

$$\mathcal{A}^W_\rho = \{\mathrm{rep}_W(\mathcal{R}, \dot{I}) \mid (\mathcal{R}, \dot{I}) \text{ is } \rho\text{-allowable over } W\} \subseteq \mathcal{C}^W.$$

By Lemma 3.16, the sequence $\langle \mathcal{A}^W_\rho \mid \rho \in \mathrm{Ord}\rangle$ is decreasing, and at limit stages it is defined by intersection. Since all terms are subsets of the single set $\mathcal{C}^W$, the sequence stabilizes.

Definition 3.20 (The stabilized level and ∞-allowability)**.** *Let α_0 be the least ordinal α such that*

$$\mathcal{A}^W_\beta = \mathcal{A}^W_\alpha \qquad \text{for every } \beta \geqslant \alpha.$$

Set

$$\alpha^* = \alpha_0 + 1.$$

A pair $(\mathcal{R}, \dot{I})$, consisting of a local hybrid presentation over W and its auxiliary name, is called ∞-allowable over W *if*

$$\mathrm{rep}_W(\mathcal{R}, \dot{I}) \in \mathcal{A}^W_{\alpha_0}.$$

Equivalently,

$$\mathrm{rep}_W(\mathcal{R}, \dot{I}) \in \mathcal{A}^W_{\alpha^*},$$

since the hierarchy has stabilized at α_0.

The passage from α_0 to $\alpha^* = \alpha_0 + 1$ is only a technical convenience. The class has already stabilized at α_0, but α^* is a successor ordinal, so the successor clause of Definition 3.15 can be used literally. Thus, whenever an ∞-allowable construction reaches a uniformization tag

$$(\mathrm{UF}, \dot{x}, \dot{z}, \dot{m})$$

after an initial segment $(\mathcal{R}_\beta, \dot{I}_\beta)$, it evaluates clause (a) of Definition 3.15 with $\rho = \alpha^*$. More explicitly, writing

$$A = \operatorname{supp}_{\mathrm{loc}}((\dot{x}, \dot{m}), \mathbb{P}_\beta),$$

a candidate is a $\mathbb{P}_A$-name $\dot{y}$ for a real. In the displayed test below, $P_{\mathcal{S}}$ denotes the terminal post-branch forcing associated with the relative extension $\mathcal{S}$. The candidate is accepted at rank $\zeta < \alpha^*$ iff

$$\mathbb{P}_\beta \Vdash (\dot{x}, \dot{y}^{\uparrow\beta}) \in A_{\dot{m}}$$

and

$$\begin{aligned} &\forall(\mathcal{S}, \dot{J})\big((\mathcal{S}, \dot{J}) \rhd_\zeta (\mathcal{R}_\beta, \dot{I}_\beta) \\ &\qquad\Rightarrow \mathbb{P}_\beta \Vdash \text{“there is no } p \in P_{\mathcal{S}}/\dot{G}_\beta \text{ such that} \\ &\qquad\qquad\qquad p \Vdash (\dot{x}^{\uparrow\mathcal{S}}, \dot{y}^{\uparrow\mathcal{S}}) \notin A_{\dot{m}^{\uparrow\mathcal{S}}}\text{”}\big). \end{aligned}$$

If such candidates exist, the construction chooses the least possible ζ, and then the $<_{M_1}$-least localized presentation of such a name $\dot{y}$ in the sense of Definition 3.5. If no candidate satisfies the displayed preservation test, clause (b) of Definition 3.15 is used.

Countable local killing tails. We shall use the following consequence of the local-support convention. Suppose that $(\mathcal{R}, \dot{I})$ is a ρ-allowable pair over W. Let $\mathbb{P}$ be the terminal post-branch forcing associated with $\mathcal{R}$. Suppose that $\dot{x}, \dot{y}$ are $\mathbb{P}$-names for reals, and that $\dot{m}$ is a $\mathbb{P}$-name for a natural number. If there is a relative ρ-allowable extension

$$(\mathcal{S}, \dot{J}) \rhd_\rho (\mathcal{R}, \dot{I})$$

and a quotient condition p in the associated quotient which forces

$$(\dot{x}^{\uparrow\mathcal{S}}, \dot{y}^{\uparrow\mathcal{S}}) \notin A_{\dot{m}^{\uparrow\mathcal{S}}},$$

then such a witness may be chosen so that $\mathcal{S}$ is obtained from $\mathcal{R}$ by appending a countable local hybrid tail. Indeed, write the complement of A_m as $\exists r\, \theta(r, x, y, m)$, with θ a Π^1_3 formula. Strengthen the quotient condition, if necessary, and use the maximum principle to choose a quotient name $\dot{r}$ such that the strengthened condition forces $\theta(\dot{r}, \dot{x}^{\uparrow\mathcal{S}}, \dot{y}^{\uparrow\mathcal{S}}, \dot{m}^{\uparrow\mathcal{S}})$. The condition and the names $\dot{x}, \dot{y}, \dot{m}, \dot{r}$ are read on a countable admissible support in the appended tail, after closing under the local supports appearing in regularity-base witnesses and under the names coded at coding stages. The regularity stages in this restricted tail keep their lifted footprint names and still use the forcings computed in their witnessed bases. The induced complete subpresentation is a countable local hybrid tail; by Lemmas 3.17 and 3.16, it is still relative ρ-allowable. Completeness preserves the displayed forcing statement.

For later reference we also isolate the intentional uses of the coding machinery. Let $(\mathcal{R}, \dot{I})$ be a ρ-allowable pair over W, witnessed by a bookkeeping function F. Let $\mathbb{P}$ be the terminal post-branch forcing associated with $\mathcal{R}$, and let $G \subseteq \mathbb{P}$ be generic. We write

$$\mathrm{Int}_\Phi(\mathcal{R}, F, G)$$

for the set of reals intentionally submitted to the coding machinery by the presentation. Thus, if a direct coding stage uses a name $\dot{w}$, the corresponding element is $\dot{w}^{G_\beta}$. If a well-order stage uses a tag $(\dot{z}_0, \dot{z}_1)$, let (A_i, σ_i) be the $<_{M_1}$-least localized presentation of $\dot{z}_i$, for $i < 2$. If these presentations are distinct, the stage contributes the well-order tag with the smaller presentation placed first. If a uniformization stage is used, the stage contributes the real

$$\langle \mathrm{UF}, x, u, m \rangle$$

selected by clause (a) or clause (b) of Definition 3.15. With this notation,

$$w \in \mathrm{Int}_\Phi(\mathcal{R}, F, G) \quad \Longrightarrow \quad \Phi(w).$$

Conversely, Lemma 3.12 applies to the underlying locally allowable coding presentation: in an allowable extension, every new instance of $\Phi(w)$ is produced at an intentional coding stage, except for instances already present in the previous initial segment.

4 The final iteration

We now define the forcing which will give the final model. The definition is the local M_1-analogue of the last iteration in the Π^1_3-uniformization and Δ^1_3-well-order construction. The iteration uses the stabilized class of ∞-allowable local hybrid presentations from Definition 3.20. The bookkeeping places ordinary Cohen stages cofinally often and, for every finite tuple of real names, places the corresponding relative Random, measure-amoeba, and category-amoeba stages cofinally often over the associated base. The bases used in the M_1-case are precisely the finite-real-parameter models

$$L[T_2, a_0, \ldots, a_{k-1}]$$

whose real parameters are read on countable admissible supports. In particular, the one-parameter bases $L[T_2, a]$ are included. The stages over these bases yield the covering statement used with the Martin–Solovay tree T_2 in Hjorth's proof of Lebesgue measurability and the Baire property.

Throughout this section, ω_2 denotes the ordinal $\omega_2^{M_1}$, equivalently the same ω_2 in the preliminary branch extension W. We fix, in M_1, a bookkeeping function

$$F : \omega_2 \longrightarrow H^{M_1}_{\omega_2}$$

with the following properties. The values of F are decoded relative to the current associated hybrid forcing. There is a cofinal subset of ω_2 on which F is the ordinary Cohen tag. Moreover, whenever finitely many real names over a countable admissible support determine a regularity stage with canonical admissibility witness

$$(A_{\mathrm{rb}}, \dot{W}_{\mathrm{rb}}, B^0_{\mathrm{rb}}, \dot{B}_{\mathrm{rb}}),$$

where

$$\dot{W}_{\mathrm{rb}} = L[T_2, \dot{a}_0, \ldots, \dot{a}_{k-1}],$$

each of the corresponding finite-real-parameter tags

$$(\mathrm{Rand}; \dot{a}^{\uparrow}_0, \ldots, \dot{a}^{\uparrow}_{k-1}), \qquad (\mathrm{Am}_{\mathcal{N}}; \dot{a}^{\uparrow}_0, \ldots, \dot{a}^{\uparrow}_{k-1}), \qquad (\mathrm{Am}_{\mathcal{M}}; \dot{a}^{\uparrow}_0, \ldots, \dot{a}^{\uparrow}_{k-1})$$

appears cofinally often after the stage at which the tuple becomes meaningful. Thus the regularity bookkeeping ranges over the allowed bases of the form $L[T_2, \vec{a}]$, not over unspecified named models. The forcing at such a stage is the one computed in the lifted base generated by the displayed real names. The witness records the support on which the real parameters $\dot{a}_i$ are read, the countable branch-product support B^0_{rb} of the W-parameters occurring in those names, and the branch footprint $\dot{B}_{\mathrm{rb}}$. Finally, whenever an M_1-description becomes meaningful over some initial segment as one of the following objects,

$$(\mathrm{WO}, \dot{z}_0, \dot{z}_1), \qquad (\mathrm{UF}, \dot{x}, \dot{y}, \dot{m}),$$

where the displayed symbols are names for reals, and $\dot{m}$ is a name for a natural number, that code appears cofinally often after the stage at which it becomes meaningful. We use disjoint recursive real codings for the two kinds of coding actions and write

$$\langle \mathrm{WO}, z_0, z_1 \rangle, \qquad \langle \mathrm{UF}, x, y, m \rangle$$

for the corresponding tagged reals. Thus the well-order coding and the uniformization coding are syntactically disjoint.

Definition 4.1 (The main iteration)**.** *We define, in W, an increasing sequence*

$$\langle (\mathcal{R}_\beta, \dot{I}_\beta) \mid \beta \leqslant \omega_2 \rangle$$

of local hybrid presentations in W and auxiliary names. The subscript β denotes the outer bookkeeping stage. In the second uniformization case below, the extension appended at the stage is required to be a countable local hybrid tail over the current presentation. Thus each successor step adds either one hybrid coordinate or a countable block of hybrid coordinates. Since ω_2 is regular, every $\mathcal{R}_\beta$ is again a single local hybrid presentation of length at most ω_2.

Set $\mathcal{R}_0$ to be the trivial post-branch presentation, so that

$$\mathbb{P}_0 = \mathbf{1},$$

and the corresponding full forcing over M_1 is simply Br. Let $\dot I_0$ be the canonical name for the empty set. At a limit stage $\lambda \leqslant \omega_2$, let $\mathcal{R}_\lambda$ be the mixed-support direct limit of the earlier presentations, with countable support on reservoir coordinates and finite support on the c.c.c. real-adding coordinates, and let $\dot I_\lambda$ be the canonical name for the union of the earlier interpreted auxiliary names.

Assume that $\beta < \omega_2$ and that $(\mathcal{R}_\beta, \dot I_\beta)$ has been constructed. Let $\mathbb{P}_\beta$ be the post-branch forcing associated with $\mathcal{R}_\beta$. The next step is chosen by the following cases.

Relative regularity and Cohen stages. *In these cases the hybrid successor step first adds a fresh reservoir coordinate. Thus choose a fresh copy $\mathbb{C}_\beta$ of $\mathbb{C}_{M_1}$, let $\dot g_\beta$ be its generic, and put*

$$\widehat{\mathbb{P}}_{\beta+1} = \mathbb{P}_\beta \times \mathbb{C}_\beta.$$

If $F(\beta)$ is one of the relative regularity tags

$$(\mathrm{Rand}; \dot a_0^\beta, \dots, \dot a_{k_\beta-1}^\beta), \qquad (\mathrm{Am}_{\mathcal{N}}; \dot a_0^\beta, \dots, \dot a_{k_\beta-1}^\beta), \qquad (\mathrm{Am}_{\mathcal{M}}; \dot a_0^\beta, \dots, \dot a_{k_\beta-1}^\beta),$$

and this stage has a canonical admissibility witness in the sense of Definition 3.6, let

$$(A_\beta^{\mathrm{reg}}, \dot W_\beta^{\mathrm{reg}}, B_\beta^{0,\mathrm{reg}}, \dot B_\beta^{\mathrm{reg}})$$

be that witness. Let $\dot W_\beta^{\mathrm{reg},+}$ be the corresponding $\widehat{\mathbb{P}}_{\beta+1}$-name for the lifted base. Thus

$$\dot W_\beta^{\mathrm{reg},+} = L[T_2, (\dot a_0^{\beta,\mathrm{reg}})^\uparrow, \dots, (\dot a_{k_\beta-1}^{\beta,\mathrm{reg}})^\uparrow]$$

for the least tuple of names reading the displayed real parameters. Let $\dot{\mathbb{Q}}_\beta$ be the member of the following list corresponding to the tag:

$$\dot{\mathbb{B}}_{\mathrm{rand}}^{\dot W_\beta^{\mathrm{reg},+}}, \qquad \dot{\mathbb{A}}_{\mathcal{N}}^{\dot W_\beta^{\mathrm{reg},+}}, \qquad \dot{\mathbb{A}}_{\mathcal{M}}^{\dot W_\beta^{\mathrm{reg},+}}.$$

Then

$$\mathbb{P}_{\beta+1} = \widehat{\mathbb{P}}_{\beta+1} * \dot{\mathbb{Q}}_\beta.$$

If the displayed regularity tag has no admissibility witness, the second-step forcing is trivial and we set

$$\mathbb{P}_{\beta+1} = \widehat{\mathbb{P}}_{\beta+1}.$$

If $F(\beta)$ is the ordinary Cohen tag, the second-step forcing is ordinary Cohen forcing on ω:

$$\mathbb{P}_{\beta+1} = \widehat{\mathbb{P}}_{\beta+1} * \dot{\mathbb{C}}_\omega.$$

In all these cases $\dot{I}_\beta$ is lifted canonically. Ordinary Cohen forcing is distinct from the M_1-Cohen reservoir coordinate attached at the beginning of the hybrid successor step.

Well-order stages. *Suppose that $F(\beta)$ decodes to*

$$(\mathrm{WO}, \dot{z}_0, \dot{z}_1),$$

where $\dot{z}_0$ and $\dot{z}_1$ are $\mathbb{P}_\beta$-names for reals. Let (a_i, σ_i) be the canonical localized presentation of $\dot{z}_i$, for $i < 2$. As in every hybrid successor step, this case uses the fresh reservoir coordinate attached at the beginning of the stage. If the two canonical localized presentations are equal, the second-step forcing over that reservoir coordinate is trivial and the auxiliary name is lifted. Otherwise compare them by $<_{M_1}$. If

$$(a_0, \sigma_0) <_{M_1} (a_1, \sigma_1),$$

use the second-step coding forcing for

$$\mathrm{Code}(\langle \mathrm{WO}, \dot{z}_0, \dot{z}_1 \rangle).$$

If

$$(a_1, \sigma_1) <_{M_1} (a_0, \sigma_0),$$

use the second-step coding forcing for

$$\mathrm{Code}(\langle \mathrm{WO}, \dot{z}_1, \dot{z}_0 \rangle).$$

In either nontrivial case this means, as in Definition 3.8, that the nontrivial second step over the reservoir coordinate already attached at this stage is the Jensen–Solovay almost-disjoint coding of the corresponding reshaped David code into a real. The auxiliary name is lifted unchanged.

Uniformization stages. *Suppose that $F(\beta)$ decodes to*

$$(\mathrm{UF}, \dot{x}, \dot{y}, \dot{m}),$$

where $\dot{x}$ and $\dot{y}$ are $\mathbb{P}_\beta$-names for reals and $\dot{m}$ is a $\mathbb{P}_\beta$-name for a natural number. Let

$$L_\beta = \mathrm{supp}_{\mathrm{loc}}((\dot{x}, \dot{m}), \mathbb{P}_\beta).$$

We apply the stabilized α^*-test from Definition 3.20. Thus we ask whether there are an ordinal $\zeta < \alpha^*$ and a $\mathbb{P}_{L_\beta}$-name $\dot{y}_0$ for a real such that, after canonically lifting $\dot{y}_0$ to a $\mathbb{P}_\beta$-name,

$$\mathbb{P}_\beta \Vdash (\dot{x}, \dot{y}_0^{\uparrow\beta}) \in A_{\dot{m}},$$

and no relative ζ-allowable extension in W can force this pair out of the relevant section. Explicitly, for every

$$(\mathcal{S}, \dot{J}) \rhd_\zeta (\mathcal{R}_\beta, \dot{I}_\beta)$$

in W, write $P_{\mathcal{S}}$ for the terminal post-branch forcing associated with $\mathcal{S}$. If the displayed names are lifted to $P_{\mathcal{S}}$, then

$$\mathbb{P}_\beta \Vdash \text{“}\textit{there is no condition in } P_{\mathcal{S}}/\dot{G}_\beta \textit{ forcing } (\dot{x}^{\upharpoonright\mathcal{S}}, \dot{y}_0^{\upharpoonright\mathcal{S}}) \notin A_{\dot{m}^{\upharpoonright\mathcal{S}}}.\text{”}$$

If such witnesses exist, choose the least possible ζ and then the $<_{M_1}$-least localized presentation of a witnessing $\dot{y}_0$.

In this first case, $\dot{y}_0$ is put into the tentative uniformizing set with rank ζ. We do not code $\dot{y}_0$ itself. Instead choose the $<_{M_1}$-least usable real name $\dot{u}$ such that

$$\mathbb{P}_\beta \Vdash \dot{u} \neq \dot{y}_0^{\upharpoonright\beta}$$

and such that the tagged name

$$\langle \mathrm{UF}, \dot{x}, \dot{u}, \dot{m} \rangle$$

has not already been intentionally coded at an earlier uniformization stage. If the bookkeeping guess $\dot{y}$ is already forced to be different from $\dot{y}_0^{\upharpoonright\beta}$ and has not yet been so coded, we take $\dot{u} = \dot{y}$. Now perform

$$\mathrm{Code}(\langle \mathrm{UF}, \dot{x}, \dot{u}, \dot{m} \rangle).$$

The new auxiliary name is the canonical $\mathbb{P}_{\beta+1}$-name forced to be

$$\dot{I}_\beta^{\dot{G}_\beta} \cup \{(\dot{x}^{\dot{G}_\beta}, (\dot{y}_0^{\upharpoonright\beta})^{\dot{G}_\beta}, \dot{m}^{\dot{G}_\beta}, \check{\zeta})\}.$$

If the stabilized test fails, we use the other side of the ∞-allowable dichotomy. Choose the $<_{M_1}$-least description of a countable relative α^*-allowable local hybrid tail

$$(\mathcal{S}, \dot{J}) \trianglerighteq_{\alpha*} (\mathcal{R}_\beta, \dot{I}_\beta)$$

together with a quotient condition witnessing that the current bookkeeping candidate can be forced out of the section. Writing $P_{\mathcal{S}}$ for the terminal post-branch forcing associated with $\mathcal{S}$, after restricting the quotient below that condition we have

$$P_{\mathcal{S}}/\dot{G}_\beta \Vdash (\dot{x}^{\upharpoonright\mathcal{S}}, \dot{y}^{\upharpoonright\mathcal{S}}) \notin A_{\dot{m}^{\upharpoonright\mathcal{S}}}.$$

The preceding countable-tail observation ensures that such a countable witness exists whenever any relative allowable tail can kill the candidate. We replace $\mathcal{S}$ by this canonical restricted countable tail and set

$$(\mathcal{R}_{\beta+1}, \dot{I}_{\beta+1}) = (\mathcal{S}, \dot{J}).$$

Thus the stage does not merely remember that some condition could kill the candidate; the appended countable quotient itself forces the negation.

If the value of $F(\beta)$ is not meaningful over the current initial segment, the hybrid successor step still attaches its fresh reservoir coordinate, but the second-step forcing is trivial and $\dot{I}_\beta$ is lifted canonically.

Let

$$\mathbb{P}^* = \mathbb{P}_{\omega_2}, \qquad \dot{I}^* = \dot{I}_{\omega_2}.$$

If $G^* \subseteq \mathbb{P}^*$ is W-generic, we write

$$W^* = W[G^*], \qquad I^* = (\dot{I}^*)^{G^*}.$$

Lemma 4.2 (The final presentation is ∞-allowable)**.** *The pair $(\mathcal{R}_{\omega_2}, \dot{I}_{\omega_2})$ belongs to W and is ∞-allowable. More precisely, there is an M_1-bookkeeping function F^* such that every initial segment of the final presentation is α^*-allowable relative to F^*. In addition, ordinary Cohen forcing occurs cofinally often. If a finite-real-parameter regularity stage becomes meaningful, with canonical support witness $(A_{\mathrm{rb}}, \dot{W}_{\mathrm{rb}}, B^0_{\mathrm{rb}}, \dot{B}_{\mathrm{rb}})$, then the relative Random forcing, measure-amoeba forcing, and category-amoeba forcing computed in $\dot{W}_{\mathrm{rb}}$ occur cofinally often in the final presentation.*

Proof. We argue by induction on the construction stages. At stage 0, the presentation is the trivial post-branch presentation, which is allowable by Lemma 3.9. At limit stages, the construction takes the countable-support direct limit of the earlier presentations. This is allowed by the limit clause in the derivative hierarchy defining α-allowability.

Consider a successor stage. The relative regularity cases, the ordinary Cohen case, and the well-order case are instances of the successor clauses in the local allowable recursion. In each of these cases the construction first adds the fresh M_1-Cohen reservoir coordinate and then performs the specified second step, or the trivial second step if the relevant bookkeeping value is not meaningful. Therefore the resulting presentation remains α^*-allowable over the preceding initial segment.

The first uniformization case is the stabilized successor clause from Definition 3.20. The chosen tentative value is added to the auxiliary name with its rank, and the construction intentionally codes only a competing reserved tag. Thus the new presentation is again α^*-allowable over the preceding initial segment.

It remains to check the second uniformization case. In this case the construction starts with a relative α^*-allowable continuation

$$(\mathcal{S}, \dot{J}) \trianglerighteq_{\alpha^*} (\mathcal{R}_\beta, \dot{I}_\beta),$$

a quotient condition $q \in P_{\mathcal{S}}/\dot{G}_\beta$, where $P_{\mathcal{S}}$ is the terminal post-branch forcing associated with $\mathcal{S}$, and a name $\dot{r}$ such that

$$q \Vdash \theta(\dot{r}, \dot{x}^{\uparrow\mathcal{S}}, \dot{y}^{\uparrow\mathcal{S}}, \dot{m}^{\uparrow\mathcal{S}}).$$

By the local-support convention for hybrid presentations, the condition q, the name $\dot{r}$, the lifted names

$$\dot{x}^{\uparrow\mathcal{S}}, \qquad \dot{y}^{\uparrow\mathcal{S}}, \qquad \dot{m}^{\uparrow\mathcal{S}},$$

and the data needed to interpret these objects are contained in a countable localized subpresentation of $(\mathcal{S}, \dot{J})$ over $(\mathcal{R}_\beta, \dot{I}_\beta)$. The construction chooses the $<_{M_1}$-least such support and restricts the corresponding presentation below the witnessing condition q. The resulting presentation

$$(\mathcal{S}^B, \dot{j}^B) \rhd_{\alpha*} (\mathcal{R}_\beta, \dot{I}_\beta)$$

is still relative α^*-allowable, because the derivative clauses are closed under passing to localized subpresentations and under restricting below a condition. Moreover its quotient forces

$$(\dot{x}^{\uparrow \mathcal{S}^B}, \dot{y}^{\uparrow \mathcal{S}^B}) \notin A_{\dot{m}^{\uparrow \mathcal{S}^B}}.$$

Thus the second uniformization case also preserves α^*-allowability.

It remains to record the bookkeeping. Whenever the construction appends a countable presentation in the second uniformization case, we use the bookkeeping function witnessing that chosen presentation on the newly added coordinates. On the coordinates coming from the original stages we use the original bookkeeping function F. Since ω_2 is regular, the concatenation of ω_2-many countable presentations has length at most ω_2, and hence can be indexed by ω_2 in the order in which it is built. This gives a single M_1-bookkeeping function F^* for the final presentation.

The cofinality requirements are inherited from F. The original bookkeeping function places ordinary Cohen tags cofinally often. Once a finite tuple of real names becomes meaningful as an admissibly based regularity stage, with witness $(A_{\mathrm{rb}}, \dot{W}_{\mathrm{rb}}, B^0_{\mathrm{rb}}, \dot{B}_{\mathrm{rb}})$, F places the three corresponding finite-real-parameter tags cofinally often, using the Random, measure-amoeba, and category-amoeba forcings computed in the lifted base generated by that witness. Adding countably many coordinates at individual stages does not destroy cofinality in the final indexing of length ω_2. Therefore the final presentation has all the required cofinal regularity and Cohen stages, and every initial segment is α^*-allowable relative to F^*. Hence $(\mathcal{R}_{\omega_2}, \dot{I}_{\omega_2})$ is ∞-allowable. $\square$

The next step will be to prove the final dichotomy. For each m and x, either I^* contains a tentative value (x, y, m, ξ) of least rank, in which case Lemma 3.19 keeps (x, y) in A_m through all further ∞-allowable tails, or else the second uniformization case is applied cofinally often to every candidate name for a value in the x-section and the final model forces that section of A_m to be empty.

Before proving the dichotomy we isolate the exactness property of the reserved tags used below.

Lemma 4.3 (Exactness for reserved tags)**.** *Let*

$$w = \langle \mathrm{UF}, x, y, m \rangle \quad \textit{or} \quad w = \langle \mathrm{WO}, z_0, z_1 \rangle$$

be a real of one of the reserved tagged forms in W^**. If*

$$W^* \models \Phi(w),$$

then this instance of $\Phi(w)$ *was introduced by an intentional coding step of the construction. More precisely, a code for a real of the form* $\langle \mathrm{UF}, x, y, m\rangle$ *is introduced only by a uniformization coding step, and a code for a real of the form* $\langle \mathrm{WO}, z_0, z_1\rangle$ *is introduced only by a well-order coding step.*

Proof. Fix a real $r \in W^*$ witnessing $\Phi(w)$. Thus $\Psi(r, w)$ holds in W^*, where Ψ is the predicate used in the definition of Φ. By the local-support property of the hybrid presentations, the real w, the witness r, and the data needed to verify $\Psi(r, w)$ are read in a countable localized subpresentation of the final construction over W. Closing this support under the local data used by the relevant names gives a 0-allowable local hybrid presentation $\mathcal{R}_A$ over W and a generic filter G_A such that

$$r, w \in W[G_A] \quad \text{and} \quad W[G_A] \models \Psi(r, w).$$

Hence

$$W[G_A] \models \Phi(w).$$

Apply Lemma 3.12 to this localized presentation. There is an explicit coding coordinate η of $\mathcal{R}_A$ and a name $\dot{u}_\eta$ such that

$$w = (\dot{u}_\eta)^{G_A}.$$

Thus every occurrence of $\Phi(w)$ for a reserved tag is produced by an explicit coding coordinate of the actual local presentation which reads the witness.

It remains only to identify which coding coordinates may write reserved tags. The construction uses the tags UF and WO only for the reserved uniformization and well-order decisions. Direct coding clauses do not use reals whose outer tag is UF or WO. The same convention is part of every countable localized presentation appended in the second uniformization case. Therefore an explicit coding coordinate which writes a real of the form $\langle \mathrm{WO}, z_0, z_1\rangle$ must be a well-order coding step, and an explicit coding coordinate which writes a real of the form $\langle \mathrm{UF}, x, y, m\rangle$ must be a uniformization coding step.

This proves the claimed exactness of the reserved tags. □

Lemma 4.4 (Final dichotomy for the uniformization stages)**.** *In* W^**, for every real* $x \in 2^\omega$ *and every* $m \in \omega$*, exactly one of the following alternatives holds.*

1. *There are a real* y *and an ordinal* $\xi < \alpha^*$ *such that*

$$(x, y, m, \xi) \in I^*.$$

In this case there is a unique real y_0 such that

$$W^* \models (x, y_0) \in A_m \quad \textit{and} \quad \neg\Phi(\langle \mathrm{UF}, x, y_0, m\rangle).$$

Moreover, if $y \neq y_0$ and $(x, y) \in A_m$, then

$$W^* \models \Phi(\langle \mathrm{UF}, x, y, m\rangle).$$

2. *For every real y and every ordinal $\xi < \alpha^*$,*

$$(x, y, m, \xi) \notin I^*.$$

In this case

$$W^* \models A_{m,x} = \varnothing.$$

Here $\langle \mathrm{UF}, x, y, m\rangle$ denotes the fixed recursive real coding of the tagged quadruple used at the uniformization coding stages.

Proof. The two alternatives are mutually exclusive and exhaustive by their first displayed clauses. We prove the corresponding conclusions.

Assume first that

$$X_{x,m} = \{(y, \xi) \mid \xi < \alpha^* \text{ and } (x, y, m, \xi) \in I^*\}$$

is nonempty. Let ξ_0 be the least ordinal such that $(y, \xi_0) \in X_{x,m}$ for some y. Among all names whose interpretation gives such a tuple of rank ξ_0, choose the one with $<_{M_1}$-least localized presentation. Let its value be y_0, and let $\beta_0 < \omega_2$ be the first stage at which the corresponding tuple

$$(x, y_0, m, \xi_0)$$

is inserted into the auxiliary name. By Lemma 3.19, applied to the tail of the final ∞-allowable presentation after β_0,

$$W^* \models (x, y_0) \in A_m. \tag{1}$$

We show that $\Phi(\langle \mathrm{UF}, x, y_0, m\rangle)$ fails in W^*. Suppose otherwise. By Lemma 4.3, there is an intentional uniformization coding stage $\eta < \omega_2$ which codes

$$w_0 = \langle \mathrm{UF}, x, y_0, m\rangle.$$

Choose η least with this property.

First suppose that $\eta \geqslant \beta_0$. At every stage $\eta' \geqslant \beta_0$, the lift of the name for y_0 is still a rank ξ_0 tentative value, by Lemma 3.19. If the construction at η selected some value different from y_0, then that value would have either smaller rank than ξ_0, or rank ξ_0 and a strictly earlier localized presentation. Its tuple would belong to I^*, contradicting the choice of (ξ_0, y_0). Hence y_0

is the selected value at η. The first uniformization clause never codes the selected value itself, so w_0 is not coded at η, a contradiction.

Now suppose that $\eta < \beta_0$. If w_0 is coded at η by the second uniformization case, then there are a relative ξ_0-allowable continuation $(\mathcal{S}, \dot{J})$ over $(\mathcal{R}_\eta, \dot{I}_\eta)$, a quotient condition q, and a name $\dot{r}$, all contained in the countable localized subpresentation chosen at stage η, such that, writing $P_{\mathcal{S}}$ for the terminal post-branch forcing associated with $\mathcal{S}$,

$$q \Vdash_{P_{\mathcal{S}}/\dot{G}_\eta} \theta(\dot{r}, \dot{x}^{\uparrow \mathcal{S}}, \dot{y}_0^{\uparrow \mathcal{S}}, \check{m}),$$

where $(u, v) \notin A_m$ is written as $\exists r\, \theta(r, u, v, m)$ with $\theta \in \Pi^1_3$. By Lemma 3.18 and Lemma 3.17, composing this continuation with the actual interval from η to β_0 gives a relative ξ_0-allowable continuation over $(\mathcal{R}_{\beta_0}, \dot{I}_{\beta_0})$ which forces the lift of $(\dot{x}, \dot{y}_0)$ out of A_m. This contradicts the fact that y_0 is inserted at stage β_0 as a rank ξ_0 tentative value.

Thus stage η used the first uniformization case. It selected some value y_η and coded y_0 as a different candidate. If a relative ξ_0-allowable continuation over $(\mathcal{R}_\eta, \dot{I}_\eta)$ forced the lift of $(\dot{x}, \dot{y}_0)$ out of A_m, then the same composition argument would contradict the insertion of (x, y_0, m, ξ_0) at β_0. Hence y_0 was already eligible at η as a rank ξ_0 tentative value. Since the construction chooses first the least rank and then the $<_{M_1}$-least localized presentation, the selected value y_η has rank below ξ_0, or has rank ξ_0 with an earlier localized presentation. Then the tuple for y_η belongs to I^*, again contradicting the choice of (ξ_0, y_0). Therefore

$$W^* \models \neg\Phi(\langle \mathrm{UF}, x, y_0, m \rangle). \tag{2}$$

Let now $y \neq y_0$ and suppose that $W^* \models (x, y) \in A_m$. Choose localized names $\dot{x}, \dot{y}, \dot{y}_0$ and, by the usual mixing below a condition in G^*, arrange that

$$1 \Vdash \dot{y} \neq \dot{y}_0, \qquad \dot{x}^{G^*} = x, \quad \dot{y}^{G^*} = y, \quad \dot{y}_0^{G^*} = y_0.$$

By bookkeeping, there is a stage $\gamma > \beta_0$ with

$$F(\gamma) = (\mathrm{UF}, \dot{x}, \dot{y}, \check{m})$$

after these names are available. At stage γ, the lift of y_0 is the selected tentative value by the preceding minimality argument. If $\langle \mathrm{UF}, x, y, m \rangle$ has not already been coded, the first uniformization case codes

$$\langle \mathrm{UF}, \dot{x}, \dot{y}, \check{m} \rangle,$$

since $1 \Vdash \dot{y} \neq \dot{y}_0$. Hence, in all cases,

$$W^* \models \Phi(\langle \mathrm{UF}, x, y, m \rangle). \tag{3}$$

Equations (1)–(3) prove the first alternative.

Assume now that

$$X_{x,m} = \varnothing. \tag{4}$$

Suppose toward a contradiction that $W^* \models (x, y) \in A_m$. Choose localized names $\dot{x}, \dot{y}$ for x, y, and choose a later stage $\gamma < \omega_2$ such that

$$F(\gamma) = (\mathrm{UF}, \dot{x}, \dot{y}, \check{m}).$$

The first uniformization case cannot occur at γ, since it would add some tuple (x, y', m, ξ) with $\xi < \alpha^*$ to I^*, contradicting (4). Therefore the second uniformization case is used. Thus the construction appends a countable localized relative α^*-allowable presentation, restricted below a witnessing quotient condition q, and a name $\dot{r}$ such that

$$q \Vdash \theta(\dot{r}, \dot{x}^{\uparrow}, \dot{y}^{\uparrow}, \check{m}), \tag{5}$$

where again $(u, v) \notin A_m$ is written as $\exists r\, \theta(r, u, v, m)$ with $\theta \in \Pi^1_3$.

The presentation added at γ is an initial segment of the final presentation, and the remaining quotient is an allowable local hybrid tail. By the T_2-absoluteness used for allowable tails, the Π^1_3-statement in (5) remains true in the final extension for the same witness $r = \dot{r}^{G^*}$. Hence

$$W^* \models (x, y) \notin A_m,$$

contradicting the choice of y. Therefore $A_{m,x} = \varnothing$ in W^*, as required. □

Corollary 4.5 (The final Π^1_4-uniformizing relation)**.** *In W^*, the Π^1_4-uniformization property holds for the relations in the fixed universal list. More precisely, for each $m < \omega$ define*

$$U_m(x, y) \quad \Longleftrightarrow \quad (x, y) \in A_m \ \wedge \ \neg\Phi(\langle \mathrm{UF}, x, y, m\rangle).$$

Then U_m is a Π^1_4 relation. Moreover, for every real x, if the x-section of A_m is nonempty, then there is exactly one real y such that $U_m(x, y)$. Consequently W^* satisfies the lightface Π^1_4-uniformization property. By the usual relativization, it also satisfies the boldface $\mathbf{\Pi}^1_4$-uniformization property.

Proof. Fix $m < \omega$ and a real $x \in W^*$. If the x-section of A_m is empty, then no real y satisfies $U_m(x, y)$, by definition. Suppose therefore that $A_{m,x} \neq \varnothing$. The second alternative of Lemma 4.4 is then impossible. Hence the first alternative holds, and the lemma gives a real y_0 such that

$$W^* \models (x, y_0) \in A_m \quad \text{and} \quad W^* \models \neg\Phi(\langle \mathrm{UF}, x, y_0, m\rangle).$$

Thus $U_m(x, y_0)$ holds.

The same lemma also gives uniqueness. Namely, if $y \neq y_0$ and $(x, y) \in A_m$, then

$$W^* \models \Phi(\langle \mathrm{UF}, x, y, m\rangle),$$

so $U_m(x, y)$ fails. If $(x, y) \notin A_m$, then $U_m(x, y)$ fails already by its first conjunct. Therefore y_0 is the unique value selected by U_m on the x-section of A_m.

It remains to record the projective complexity. By the choice of the universal list, $(x, y) \in A_m$ is Π^1_4. The local coding predicate $\Phi(w)$ was defined in the coding section as a Σ^1_4 predicate. Therefore

$$\neg\Phi(\langle \mathrm{UF}, x, y, m\rangle)$$

is Π^1_4. Since the class Π^1_4 is closed under finite conjunctions, $U_m(x, y)$ is Π^1_4.

For the boldface conclusion, let $A(x, y)$ be $\Pi^1_4(a)$. Using the fixed recursive pairing of reals, regard this as a lightface Π^1_4 relation in the pair (a, x) and the value y, say as a section of some member of the universal list. Applying the preceding lightface uniformization to the relation on $((a, x), y)$ and then fixing the parameter a gives a $\Pi^1_4(a)$ uniformizing relation for A. $\square$

Lemma 4.6 (The final Δ^1_4-well-order)**.** *In W^* the reals admit a Δ^1_4-definable well-order.*

Proof. We first describe the underlying order independently of the projective definition. If $z \in W^*$ is a real, let $\pi(z)$ be the $<_{M_1}$-least localized presentation, in the final hybrid presentation, of a name whose interpretation is z. Such a presentation exists because every real in the final mixed-support hybrid iteration has a bounded local support. If $z_0 \neq z_1$, then $\pi(z_0) \neq \pi(z_1)$, since the same localized name has only one interpretation in the fixed generic extension. Hence

$$z_0 \lhd z_1 \quad\Longleftrightarrow\quad z_0 \neq z_1 \text{ and } \pi(z_0) <_{M_1} \pi(z_1)$$

defines a well-order of the reals in W^*. It is the order induced by the fixed $<_{M_1}$-well-order of the canonical localized presentations.

We claim that, for distinct reals $z_0, z_1 \in W^*$,

$$z_0 \lhd z_1 \quad\Longleftrightarrow\quad W^* \models \Phi(\langle \mathrm{WO}, z_0, z_1\rangle).$$

Assume first that $z_0 \lhd z_1$. Choose localized names $\dot{z}_0, \dot{z}_1$ whose final canonical localized presentations are $\pi(z_0)$ and $\pi(z_1)$. The supports of these presentations are bounded in ω_2, so there is an initial segment $\mathbb{P}_\beta$ over which these names are already available. By the bookkeeping property of F, there is a later stage $\gamma > \beta$ at which

$$F(\gamma) = (\mathrm{WO}, \dot{z}_0, \dot{z}_1).$$

At stage γ the well-order clause compares the same canonical localized presentations. The localization convention and the product/tail lemmas ensure that passing to the larger initial segment, or later appending a countable local tail, does not change the $<_{M_1}$-comparison of these presentations. Since $\pi(z_0) <_{M_1} \pi(z_1)$, the stage explicitly codes

$$\langle \mathrm{WO}, \dot{z}_0, \dot{z}_1 \rangle.$$

After interpreting by G^*, Lemma 3.9 yields

$$W^* \models \Phi(\langle \mathrm{WO}, z_0, z_1 \rangle).$$

Conversely suppose that $z_0 \neq z_1$ and

$$W^* \models \Phi(\langle \mathrm{WO}, z_0, z_1 \rangle).$$

By Lemma 4.3, this code was introduced at an intentional well-order coding stage of the final presentation. Thus, at some stage, the construction decoded a well-order tag, compared the canonical localized presentations of two names whose interpretations are z_0 and z_1, and coded the ordered tag with the smaller presentation placed first. Again the localization and product/tail conventions imply that this comparison agrees with the comparison of the final canonical presentations $\pi(z_0)$ and $\pi(z_1)$. Therefore $\pi(z_0) <_{M_1} \pi(z_1)$, and hence $z_0 \lhd z_1$. This proves the claim.

Now define, in W^*,

$$z_0 <_\Delta z_1 \quad \Longleftrightarrow \quad z_0 \neq z_1 \ \wedge \ \Phi(\langle \mathrm{WO}, z_0, z_1 \rangle).$$

By the claim, $<_\Delta$ is exactly $\lhd$, and hence is a well-order of the reals.

It remains to record the projective complexity. Since Φ is the local Σ^1_4-coding predicate, the displayed definition of $<_\Delta$ is Σ^1_4. The claim also gives, for distinct reals z_0, z_1, that exactly one of

$$\Phi(\langle \mathrm{WO}, z_0, z_1 \rangle), \qquad \Phi(\langle \mathrm{WO}, z_1, z_0 \rangle)$$

holds. Hence, in W^*, the same relation is equivalently defined by

$$z_0 <_\Delta z_1 \quad \Longleftrightarrow \quad z_0 \neq z_1 \ \wedge \ \neg\Phi(\langle \mathrm{WO}, z_1, z_0 \rangle).$$

This second definition is Π^1_4. Thus $<_\Delta$ is both Σ^1_4 and Π^1_4, and consequently is a Δ^1_4-definable well-order of the reals. $\square$

Lemma 4.7 (Localized covering over $L[T_2, a]$)**.** *Let $a \in W^*$ be a real and put*

$$N_a = L[T_2, a].$$

Then there is a Borel null set $Z^{\mathcal{N}}_a \in W^*$ such that

$$\bigcup \{B \mid B \text{ is a Borel null set coded in } N_a\} \subseteq Z^{\mathcal{N}}_a.$$

There is also a Borel meager set $Z_a^{\mathcal{M}} \in W^*$ *such that*

$$\bigcup\{B \mid B \textit{ is a Borel meager set coded in } N_a\} \subseteq Z_a^{\mathcal{M}}.$$

Consequently

$$R_a = \{z \in \omega^\omega : z \textit{ is not Random-generic over } N_a\}$$

is null, and

$$C_a = \{z \in \omega^\omega : z \textit{ is not Cohen-generic over } N_a\}$$

is meager.

Proof. Fix $a \in W^*$. By Lemma 3.10, choose a countable admissible support $A \subseteq \omega_2$ and a $\mathbb{P}_A$-name σ such that

$$\sigma^{G_A} = a,$$

where $G_A = G^* \cap \mathbb{P}_A$. Since ω_2 is regular, choose $\beta < \omega_2$ with $A \subseteq \beta$. Let

$$\dot{a} = (\sigma)^{\uparrow\beta}$$

and let $\dot{N}$ be the $\mathbb{P}_\beta$-name for $L[T_2, \dot{a}]$. Then the corresponding regularity tags for $\dot{N}$ are admissibly based. Indeed, this is the one-parameter instance of Definition 3.6: take

$$A_{\text{rb}} = A, \qquad k = 1, \qquad \dot{a}_0 = \sigma, \qquad \dot{W}_{\text{rb}} = L[T_2, \sigma].$$

No further presentation data are needed. The set B^0_{rb} is the countable union of the branch-product supports of the W-parameters occurring in the name σ, and $\dot{B}_{\text{rb}}$ is the corresponding branch footprint generated by B^0_{rb} and by the explicit coding coordinates in A. Thus the bookkeeping for admissible bases applies to $\dot{N}$, and it must eventually schedule the relative measure- and category-amoeba stages over this base.

By the final bookkeeping, at some stage $\lambda_{\mathcal{N}} > \beta$ the tag

$$(\text{Am}_{\mathcal{N}}, \dot{N}^{\uparrow\lambda_{\mathcal{N}}})$$

is used. At this stage the forcing is

$$\mathbb{A}_{\mathcal{N}}^{N_a},$$

computed in the interpreted base $N_a = L[T_2, a]$. Let $Z_a^{\mathcal{N}}$ be the Borel null set added by this amoeba forcing. By the definition of amoeba forcing for the null ideal,

$$B \subseteq Z_a^{\mathcal{N}}$$

for every Borel null set B coded in N_a. Later forcing preserves the Borel code for $Z_a^{\mathcal{N}}$ and the statement that it is null. Hence the first displayed inclusion holds in W^*.

The meager case is identical. The bookkeeping later uses the tag

$$(\mathrm{Am}_{\mathcal{M}}, \dot{N}^{\upharpoonright \lambda_{\mathcal{M}}})$$

for some $\lambda_{\mathcal{M}} > \beta$. The forcing $\mathbb{A}_{\mathcal{M}}^{N_a}$ adds a Borel meager set $Z_a^{\mathcal{M}}$ covering every Borel meager set coded in N_a, and this covering relation remains true in the final extension.

Finally, a real is not Random-generic over N_a iff it belongs to some Borel null set coded in N_a. Hence

$$R_a \subseteq Z_a^{\mathcal{N}},$$

so R_a is null. Similarly, a real is not Cohen-generic over N_a iff it belongs to some Borel meager set coded in N_a. Hence

$$C_a \subseteq Z_a^{\mathcal{M}},$$

so C_a is meager. □

Lemma 4.8 (Hjorth–Solovay regularity in the final model)**.** *In W^*, every boldface $\mathbf{\Sigma}_3^1$ set of reals is Lebesgue measurable and has the Baire property.*

Proof. We give the argument in the form in which it is used in Hjorth's proof of Corollary 2.4 of [3], with the use of Martin's Axiom replaced by Lemma 4.7. Let $a \in W^*$ be a real parameter and put

$$N_a = L[T_2, a].$$

Here T_2 is the Martin–Solovay tree fixed in the preliminaries. By Lemma 4.7, almost every real in W^* is Random over N_a, and comeagerly many reals in W^* are Cohen over N_a.

Let $A \subseteq \omega^\omega$ be $\Sigma_3^1(a)$, say

$$A = \{z : W^* \models \varphi(z, a)\},$$

where $\varphi(v, a)$ is a $\Sigma_3^1(a)$ formula. We prove first that A is Lebesgue measurable. Work in N_a with the Random algebra $\mathbb{B}_{\mathrm{rand}}^{N_a}$ and let $\dot{r}$ be its canonical name for the Random real. The Boolean value

$$\|\varphi(\dot{r}, a)\|_{\mathbb{B}_{\mathrm{rand}}^{N_a}}$$

is represented by a Borel set $B \subseteq \omega^\omega$, coded in N_a, modulo null sets. If z is Random over N_a, then the forcing theorem for $\mathbb{B}_{\mathrm{rand}}^{N_a}$ gives

$$z \in B \iff N_a[z] \models \varphi(z, a).$$

The Martin–Solovay tree T_2, together with its absolute complement from Corollary 2.28, gives the required Σ^1_3 generic absoluteness through all forcing extensions used here. Hence, for every z Random over N_a,

$$N_a[z] \models \varphi(z,a) \iff W^* \models \varphi(z,a).$$

Thus A and B can differ only on R_a, the set of reals which are not Random over N_a. By Lemma 4.7, R_a is null. Therefore $A \triangle B$ is null, and A is Lebesgue measurable.

The proof of the Baire property is the category analogue. Work in N_a with ordinary Cohen forcing $\mathbb{C}^{N_a}$ and let $\dot{c}$ be its canonical Cohen real. The Boolean value

$$\|\varphi(\dot{c},a)\|_{\mathbb{C}^{N_a}}$$

is represented, modulo meager sets, by a Borel set $C \subseteq \omega^\omega$ coded in N_a. If z is Cohen over N_a, then the forcing theorem gives

$$z \in C \iff N_a[z] \models \varphi(z,a).$$

The same T_2-absoluteness yields

$$N_a[z] \models \varphi(z,a) \iff W^* \models \varphi(z,a)$$

for every such Cohen real z. Hence $A \triangle C \subseteq C_a$, where C_a is the meager set of reals which are not Cohen over N_a. Therefore A differs from the Borel set C by a meager set, and A has the Baire property.

Since the parameter a was arbitrary, the conclusion holds for all boldface $\boldsymbol{\Sigma}^1_3$ sets of reals in W^*. $\square$

4.1 Σ^1_3-uniformization in small M_1-generic extensions

It remains to record the lower-level Σ-uniformization conclusion. Unlike the Π^1_4-uniformization theorem, this part does not use the coding predicate Φ. It follows from the relativized Steel capture analysis of $M_1(s)$.

What is used is the standard Steel analysis of M_1 and of its relativizations. We only need this analysis for the small generic extensions which occur in the construction. Thus, throughout this subsection, a *small generic extension of* M_1 means an extension $M_1[G]$ by a set forcing whose size in M_1 is below the Woodin cardinal of M_1 and to which the canonical iteration strategy of M_1 lifts. All intermediate models used in the construction, and in particular the final model W^*, are of this form.

For a real s in such an extension, let $M_1(s)$ denote the canonical proper class mouse over s with one Woodin cardinal. The relevant mice exist and are iterable by the lifted M_1-strategy. The argument below uses a capture theorem for this canonical mouse over s; it does not identify $M_1(s)$ with the forcing extension generated by s.

Definition 4.9. *A* simple s-mouse *is a sound, Π^1_2-iterable s-premouse which projects to ω and is an initial segment of the canonical construction of $M_1(s)$.*

The assertion that a real c codes a sound simple s-mouse is a $\Pi^1_2(s,c)$ condition. Any two such mice compare by initial segment. Hence the reals of $M_1(s)$ carry the usual good $\Sigma^1_3(s)$ well-order $<^1_s$: first compare the least simple s-mouse containing the real, and then use the canonical well-order of that mouse.

We shall use the following standard form of Steel's relativized capture theorem at the first mouse level; see the projective definability and comparison analysis of M_1 in [13, 12].

Fact 4.10. *Let $N = M_1[G]$ be a small generic extension of M_1, and let $s \in \mathbb{R}^N$.*

1. *If N satisfies a $\Sigma^1_3(s)$ statement of the form*

$$\exists u\, \exists v\, \theta(s,u,v),$$

   where θ is Π^1_2, then there are such witnesses $u, v \in M_1(s)$.

2. *For reals from $M_1(s)$, Π^1_2 truth is computed correctly by $M_1(s)$ and is preserved to N. Equivalently, if $u, v \in M_1(s)$ and θ is Π^1_2, then*

$$M_1(s) \models \theta(s,u,v) \quad \Longleftrightarrow \quad N \models \theta(s,u,v).$$

3. *The well-order $<^1_s$ is good for Σ^1_3 definitions. More explicitly, for every Π^1_2 formula $\theta(s,u,v)$, the relation*

$$\mathrm{Least}_\theta(s,u)$$

   saying that $u \in M_1(s)$ and u is the $<^1_s$-least real for which there is a $v \in M_1(s)$ with $M_1(s) \models \theta(s,u,v)$ is uniformly $\Sigma^1_3(s)$.

For completeness, let us spell out why this is the right theorem to apply. The mouse $M_1(s)$ is the canonical mouse over the parameter s. The comparison theorem for Π^1_2-iterable s-mice identifies its countable initial segments with the segments captured by the Π^1_2 mouse condition. Therefore a small generic extension of M_1 cannot create a new $\Sigma^1_3(s)$ witness without some countable initial segment of $M_1(s)$ already capturing the corresponding branch. The same comparison analysis gives the Π^1_2 correctness used in item (2), and the usual definition of the well-order by least simple s-mouse gives item (3).

Theorem 4.11. *Every small generic extension of M_1 satisfies boldface $\mathbf{\Sigma}^1_3$-uniformization. In particular, W^* satisfies Σ^1_3-uniformization.*

Proof. Let $N = M_1[G]$ be a small generic extension of M_1. Work in N, and let $A \subseteq \mathbb{R}^2$ be a boldface Σ^1_3 relation. The same argument also works for lightface Σ^1_3-relations. Fix a real parameter a and a Π^1_2 formula ψ such that

$$A(x, y) \iff \exists z\, \psi(a, x, y, z).$$

For each real x, put $s = a \oplus x$. Define $A^*(x, y)$ to hold iff

$$\mathrm{Least}_\theta(s, y),$$

where $\theta(s, y, z)$ is the Π^1_2 formula obtained from $\psi(a, x, y, z)$ after decoding $s = a \oplus x$. By Fact 4.10(3), the relation A^* is $\Sigma^1_3(a)$, hence boldface Σ^1_3.

We verify that A^* uniformizes A. If $A^*(x, y)$ holds, then by definition there is some $z \in M_1(a \oplus x)$ such that

$$M_1(a \oplus x) \models \psi(a, x, y, z).$$

By Fact 4.10(2), $N \models \psi(a, x, y, z)$. Hence $A(x, y)$ holds.

The relation A^* is single-valued because $<^1_{a\oplus x}$ is a well-order and $A^*(x, y)$ asserts that y is the $<^1_{a\oplus x}$-least real in $M_1(a \oplus x)$ for which a suitable z exists.

Finally suppose that the x-section of A is nonempty in N. Then

$$N \models \exists y \exists z\, \psi(a, x, y, z).$$

By Fact 4.10(1), there are witnesses $y, z \in M_1(a \oplus x)$. Therefore the set of $<^1_{a\oplus x}$-candidates is nonempty, so it has a least element. For this least element y_0, Fact 4.10(3) gives $A^*(x, y_0)$.

Thus A^* is a boldface Σ^1_3 uniformization of A in N. The final model W^* is a small generic extension of M_1 by the forcings fixed in the construction, so the final assertion follows. $\square$

Theorem 4.12 (The main theorem in the M_1 case). *Assume that M_1 exists. There is a forcing extension W^* such that*

$$W^* \models 2^{\aleph_0} = \aleph_2,$$

every boldface $\boldsymbol{\Sigma}^1_3$ set of reals is Lebesgue measurable and has the Baire property, the Σ^1_3- and Π^1_4-uniformization properties hold, and the reals admit a Δ^1_4-definable well-order.

Proof. Let $W^* = W[G^*]$ be the final model obtained from Definition 4.1. By Lemma 3.9 and Fact 4.2, the final forcing is a proper, $\aleph_2$-c.c. hybrid presentation of length $\omega_2 = \omega_2^{M_1}$ with iterands of size at most $\aleph_1$. Since W satisfies CH, the final forcing has size at most ω_2. Hence $W^* \models 2^{\aleph_0} \leqslant \aleph_2$. On the other hand, ordinary Cohen forcing occurs cofinally often in the final presentation, and each such nontrivial coordinate adds a new real. Therefore W^* has at least ω_2 many reals. Thus $W^* \models 2^{\aleph_0} = \aleph_2$.

Theorem 4.11 gives the boldface Σ^1_3-uniformization property. Corollary 4.5 gives the boldface Π^1_4-uniformization property. Lemma 4.6 gives a Δ^1_4-definable well-order of the reals. Finally, Lemma 4.8 gives Lebesgue measurability and the Baire property for all boldface $\boldsymbol{\Sigma}^1_3$ sets of reals. These are exactly the asserted conclusions. $\square$

5 The uniform M_n-version

The preceding sections were written in full detail for the first nontrivial case, namely for the construction over M_1. We now record the uniform form of the argument. The point of this section is not to introduce a new forcing construction, but to make explicit the replacements which turn the M_1-argument into the general M_n-argument and to isolate the few places where the shift in projective complexity is used.

Fix for the rest of this section a natural number $n \geqslant 1$, and assume that M_n exists. Let

$$\kappa_n = (\omega_2)^{M_n}.$$

We work over the canonical M_n-ground in exactly the same way as above. Thus we let

$$\vec{S}^n = \langle S^n_\xi \mid \xi < \omega_1^{M_n} \rangle$$

be the M_n-least independent sequence of Suslin trees obtained from the canonical M_n-diamond sequence of Lemma 2.20, and we first pass to the finite-support branch extension

$$W_n = M_n[\langle b_\xi \mid \xi < \omega_1^{M_n} \rangle].$$

All subsequent forcing is a hybrid coding iteration over W_n. It uses the same reservoir convention as the detailed M_1-construction. Thus every successor stage first attaches a fresh M_n-Cohen reservoir coordinate, added by the M_n-computed countably closed forcing. The second-step forcing is then a finite-support real-adding coordinate: a relative Random algebra, a relative amoeba forcing for measure or category, ordinary Cohen forcing, a Jensen–Solovay almost-disjoint coding forcing, or the trivial forcing. Random and amoeba coordinates are understood in the relative sense: the stage is tagged by an admissible base of the form $L[T_{n+1}, \vec{a}]$, with finitely many real parameters read on a countable admissible support, and uses the forcing computed in that base, not the corresponding forcing recomputed in the ambient universe. At an explicit coding coordinate, the fresh reservoir generic chooses the coding area in $\vec{S}^n$; the construction writes the relevant real into the associated ω-blocks of the Suslin sequence, reshapes in the sense of David, and finally almost-disjointly codes the reshaped set by a real.

The following dictionary will be used throughout this section:

$$M_1 \rightsquigarrow M_n, \qquad T_2 \rightsquigarrow T_{n+1},$$

$$\mathbf{\Sigma}^1_3 \rightsquigarrow \mathbf{\Sigma}^1_{n+2}, \qquad \Pi^1_4 \rightsquigarrow \Pi^1_{n+3}, \qquad \Phi \rightsquigarrow \Phi_n.$$

Here T_{n+1} is the canonical weakly homogeneous tree fixed in Definition 2.26; by Theorem 2.25 and Lemma 2.27, it represents the universal $\mathbf{\Sigma}^1_{n+2}$ set and has the required small-generic absoluteness properties. In the M_n-version, Definition 3.6 is read with M_1 replaced by M_n and with T_2 replaced by T_{n+1}. Thus a relative regularity stage is generated by finitely many real names over a countable admissible support, and its base has the form

$$L[T_{n+1}, \dot{a}_0, \ldots, \dot{a}_{k-1}].$$

Definition 5.1 (The M_n-localized coding predicate)**.** *Let $\Phi_n(r)$ be the predicate obtained from the construction of Φ in Section 3 by replacing M_1 by M_n, the M_1-least Suslin sequence by $\vec{S}^n$, and the M_1-local presentation order by the M_n-local presentation order. Thus $\Phi_n(r)$ asserts that r is decoded by a correct M_n-localized coding witness: a lower-part M_n-mouse code, a bounded localized presentation of the relevant hybrid forcing, a fresh M_n-Cohen coding area, a David-reshaped set of ordinals, and a final Jensen–Solovay almost-disjoint code.*

For the two tags used in the final construction we reserve the forms

$$\langle \mathrm{WO}, n, z_0, z_1\rangle \qquad \textit{and} \qquad \langle \mathrm{UF}, n, x, y, m\rangle.$$

These tags are not used as direct coding tags except at the corresponding well-order and uniformization stages.

Lemma 5.2 (Complexity and exactness of Φ_n)**.** *In the final M_n-construction, Φ_n is a Σ^1_{n+3} predicate. Moreover, for the reserved tags the following exactness statement holds. If*

$$W^*_n \models \Phi_n(\langle \mathrm{WO}, n, z_0, z_1\rangle),$$

then this code was introduced at an intentional well-order coding stage comparing the canonical localized presentations of z_0 and z_1. If

$$W^*_n \models \Phi_n(\langle \mathrm{UF}, n, x, y, m\rangle),$$

then this code was introduced at an intentional uniformization stage for the triple (x, y, m). Conversely, every intentional coding of either of these reserved tags gives the corresponding instance of Φ_n in the final model.

Proof. The complexity calculation is the uniform version of the calculation for Φ in the M_1 case. A witness to $\Phi_n(r)$ is a real coding a countable localized presentation together with the associated decoding data. The correctness of the lower-part mouse and of the localized M_n-initial segment is expressed using the Π_n-iterability analysis from Definition 2.11, Fact 2.12, and Lemma 2.15. Steel's definability theorem places the relevant verification one projective level above the definition of the M_n-well-order, hence at level Σ^1_{n+3}. The remaining clauses–that the displayed Suslin-tree pattern is present, that the reshaping is correct, and that the almost-disjoint code decodes the reshaped set–are arithmetic or projective of lower complexity relative to that mouse witness. Thus the whole predicate is Σ^1_{n+3}.

The exactness assertion is the M_n-version of Lemma 4.3. The proof of no unwanted codes uses independence of the Suslin sequence, freshness of the M_n-Cohen coding areas, and branch omission for selected coordinates. The branch-footprint part is unchanged: the finitely many real names generating a regularity base are read on countable admissible supports, and hence generate only countably many coding areas; the footprint records the selected coordinates coming from those areas. Hence the proof of Lemma 3.12 applies after replacing M_1 by M_n. Since the two reserved tag forms are excluded from direct coding and are used only by the well-order and uniformization clauses, any final occurrence of Φ_n on such a tag is intentional, and every intentional code is read by Φ_n. $\square$

Let

$$\langle A^n_m \mid m \in \omega \rangle$$

be a fixed universal enumeration of lightface Π^1_{n+3} subsets of $(\omega^\omega)^2$. Boldface parameters are handled, as usual, by allowing the bookkeeping to range over names for real parameters and by coding the parameter into the first coordinate. Thus it is enough to define uniformizing relations for the displayed universal family.

Definition 5.3 (M_n-ρ-allowability and M_n-∞-allowability)**.** *The classes of M_n-ρ-allowable and M_n-∞-allowable hybrid presentations are obtained from Definitions 3.8, 3.13, 3.14, 3.15, and 3.20 by making the following replacements:*

$$M_1\textit{-localized presentation} \rightsquigarrow M_n\textit{-localized presentation},$$

$$T_2 \rightsquigarrow T_{n+1}, \qquad L[T_2, \dot{a}_0, \ldots, \dot{a}_{k-1}] \rightsquigarrow L[T_{n+1}, \dot{a}_0, \ldots, \dot{a}_{k-1}],$$

$$\Phi \rightsquigarrow \Phi_n, \qquad \Pi^1_4 \rightsquigarrow \Pi^1_{n+3}, \qquad A_m \rightsquigarrow A^n_m.$$

In particular, the regularity bases in the M_n-version are precisely the finite-real-parameter models

$$L[T_{n+1}, a_0, \ldots, a_{k-1}]$$

whose real parameters are read on countable admissible supports. At a successor uniformization stage for a triple $(\dot{x}, \dot{y}, \dot{m})$ the construction asks the same stabilized question as before: is there, locally and relative to the current M_n-allowable presentation, a tentative value in the $\dot{x}$-section of $A^n_{\dot{m}}$ which cannot be removed by the relevant future M_n-allowable tails? If yes, the least such value, ordered by rank and by the M_n-canonical order of localized presentations, is protected and all competing candidates are coded with their *reserved* UF*-tags. If no, a countable relative M_n-ρ-allowable local hybrid tail is chosen which forces the candidate out of the section.*

Lemma 5.4 (Structural lemmas for M_n-allowability)**.** *The shrinking, tail, product, and persistence lemmas for allowability hold for the M_n-allowable hierarchy. More explicitly:*

1. *if a presentation is M_n-β-allowable and $\alpha < \beta$, then it is M_n-α-allowable;*
2. *tails of M_n-ρ-allowable presentations are relative M_n-ρ-allowable; moreover, any tail witnessing that a candidate can be forced out of an A^n_m-section can be replaced by a countable local hybrid tail with the same forcing property;*
3. *products of M_n-ρ-allowable presentations are again represented by M_n-ρ-allowable presentations after the usual concatenation and renaming of the hybrid supports;*
4. *tentative values remain in their Π^1_{n+3} sections through all later M_n-∞-allowable tails.*

Proof. The first three assertions are formal consequences of the recursive definition, exactly as in Lemmas 3.16, 3.17, and 3.18. The proof uses only the fact that the class at a later derivative stage has more restrictions, that the hybrid presentation can be factored into an initial part and a relative tail, that relative regularity tags carry finitely many real parameters and branch footprints under products, and that the fresh coding areas for a product can be renamed to fresh coordinates. The countable-tail refinement in (2) is the same local-support argument used in the M_1 construction: choose a name for the witness to the relevant Σ^1_{n+3} statement and restrict to the countable complete subpresentation generated by that name, the condition, and the parameters.

For (4), suppose a value has been declared tentative at a successor stage. By definition, every relevant future M_n-ρ-allowable tail preserves its membership in the relevant Π^1_{n+3} section. The only possible loss of membership would be witnessed by a Σ^1_{n+3} statement in a later extension. This statement is represented, relative to the real parameters already present, by the canonical tree T_{n+1} and its Martin–Solovay absolute complement. Lemma 2.27 therefore gives the same small-generic absoluteness used in the T_2 argument. Hence the proof of Lemma 3.19 lifts with T_2 replaced by T_{n+1}. □

Definition 5.5 (The final M_n-iteration)**.** *Let*

$$F_n : \kappa_n \longrightarrow H^{M_n}_{\kappa_n}$$

be the M_n-least bookkeeping function with the following properties. It places ordinary Cohen tags cofinally often. Moreover, whenever finitely many real *names over a countable admissible support determine a regularity stage with canonical admissibility witness*

$$(A_{\mathrm{rb}}, \dot{W}_{\mathrm{rb}}, B^0_{\mathrm{rb}}, \dot{B}_{\mathrm{rb}}),$$

where

$$\dot{W}_{\mathrm{rb}} = L[T_{n+1}, \dot{a}_0, \ldots, \dot{a}_{k-1}],$$

the corresponding finite-real-parameter tags

$$(\mathrm{Rand}; \dot{a}^{\uparrow}_0, \ldots, \dot{a}^{\uparrow}_{k-1}), \qquad (\mathrm{Am}_{\mathcal{N}}; \dot{a}^{\uparrow}_0, \ldots, \dot{a}^{\uparrow}_{k-1}), \qquad (\mathrm{Am}_{\mathcal{M}}; \dot{a}^{\uparrow}_0, \ldots, \dot{a}^{\uparrow}_{k-1})$$

appear cofinally often after that point. The forcing is computed in the lifted base generated by the displayed real names, and the witness records the support on which the real parameters are read, the countable branch-product support B^0_{rb} of their W_n-parameters, and the branch footprint $\dot{B}_{\mathrm{rb}}$. The bookkeeping also lists, cofinally often and in all possible localized presentations, well-order tags

$$(\mathrm{WO}, \dot{z}_0, \dot{z}_1)$$

and uniformization tags

$$(\mathrm{UF}, \dot{x}, \dot{y}, \dot{m}).$$

We define by recursion a hybrid presentation

$$\langle (\mathcal{R}^n_\beta, \dot{I}^n_\beta) \mid \beta \leqslant \kappa_n \rangle.$$

At every single-coordinate successor stage of this presentation we first attach the fresh M_n-Cohen reservoir coordinate. At relative regularity stages the second-step forcing is the Random or amoeba forcing computed in the lifted base $L[T_{n+1}, \vec{a}]$ generated by the displayed finite tuple of real names, and at Cohen stages the second-step forcing is the ordinary Cohen iterand. At well-order stages we compare the final canonical localized presentations of the two interpreted reals and, if they are distinct, use the second-step almost-disjoint coding forcing to intentionally code exactly one of

$$\langle \mathrm{WO}, n, z_0, z_1 \rangle, \qquad \langle \mathrm{WO}, n, z_1, z_0 \rangle,$$

namely the tag with the smaller canonical presentation first. At uniformization stages we apply the M_n-∞-allowable successor rule from Definition 5.3. Thus, if a tentative value is found, the least tentative value is protected and the non-selected candidates are coded with their tags

$$\langle \mathrm{UF}, n, x, y, m \rangle.$$

If no tentative value is found, we append the chosen countable relative M_n-∞-allowable local hybrid tail, restricted below the quotient condition which forces the current candidate out of its A^n_m-section. At limits we take the *mixed-support hybrid limit, with countable support on reservoir coordinates and finite support on the c.c.c. real-adding coordinates. The final forcing is denoted by*

$$\mathbb{P}^n_\infty = \mathbb{P}^n_{\kappa_n},$$

and if G^n_∞ *is generic, we write*

$$W^*_n = W_n[G^n_\infty], \qquad I^*_n = (\dot{I}^n_{\kappa_n})^{G^n_\infty}.$$

Fact 5.6. *The final presentation* $(\mathcal{R}^n_{\kappa_n}, \dot{I}^n_{\kappa_n})$ *is* M_n*-*∞*-allowable. Moreover ordinary Cohen forcing occurs cofinally often. If a finite-real-parameter regularity stage becomes meaningful, with canonical support witness*

$$(A_{\mathrm{rb}}, \dot{W}_{\mathrm{rb}}, B^0_{\mathrm{rb}}, \dot{B}_{\mathrm{rb}}),$$

then the relative Random forcing, measure-amoeba forcing, and category-amoeba forcing computed in the lifted base generated by that witness occur cofinally often in the final presentation.

Proof. This is the same countable-tail argument as in Fact 4.2. The bookkeeping function F_n supplies cofinally many ordinary Cohen coordinates, cofinally many relative Random and amoeba coordinates for every finite-real-parameter regularity stage once it becomes meaningful, and cofinally many names for the well-order and uniformization tasks. At a relative regularity stage the forcing is the one computed in the witnessed base $L[T_{n+1}, \vec{a}]$. The witness carries the support on which the finitely many real parameters $\vec{a}$ are read, the countable branch-product support of the W_n-parameters occurring in those names, and the branch footprint through later tails and products. A second-case uniformization stage appends a countable relative M_n-∞-allowable local hybrid tail rather than a single coordinate. Since κ_n is regular, inserting countable blocks at κ_n many stages still gives a presentation of length at most κ_n, and the witnessing bookkeeping functions for the countable tails can be concatenated with the original bookkeeping into one global bookkeeping function. The M_n-allowability clauses are preserved by this countable-tail construction. □

Lemma 5.7 (The M_n final dichotomy)**.** *Let* $x \in \omega^\omega$*,* $m \in \omega$*, and work in* W^*_n*. Exactly one of the following alternatives holds.*

1. *The final auxiliary object contains a tentative value for the pair* (x, m)*. In this case there is a rank-and-presentation least such value* y_0*, and*

   $$(x, y_0) \in A^n_m.$$

   Moreover, for every y*,*

   $$(x, y) \in A^n_m \text{ and } y \neq y_0 \implies \Phi_n(\langle \mathrm{UF}, n, x, y, m\rangle).$$

   Also

   $$\neg\Phi_n(\langle \mathrm{UF}, n, x, y_0, m\rangle).$$

2. *The final auxiliary object contains no tentative value for (x, m). In this case*
$$A^n_m(x, \cdot) = \varnothing.$$

Proof. The proof is the proof of Lemma 4.4, with the complexity shifted by $n - 1$. We indicate the points where something has to be checked.

Suppose first that a tentative value occurs. By Lemma 5.4(4), tentative values remain in their Π^1_{n+3} sections through later M_n-∞-allowable tails. Let y_0 be the least tentative value according to the fixed rank and M_n-localized-presentation order. The bookkeeping revisits every later candidate y in the same section. Whenever $y \neq y_0$ is still a member of the x-section, the uniformization clause intentionally codes the tag $\langle \mathrm{UF}, n, x, y, m \rangle$. By Lemma 5.2, these intentional codes are exactly the final instances of Φ_n on reserved UF-tags. The selected value y_0 is never coded with its own UF-tag, again by the exactness lemma. This gives the first alternative.

Now suppose that no tentative value occurs. Let y be any real in the final model and choose a bounded localized name for y. By the cofinality of the bookkeeping, some later uniformization stage considers the corresponding triple (x, y, m). Since no tentative value is available, the construction uses the second case and appends a countable relative M_n-∞-allowable local hybrid tail, restricted below a quotient condition forcing

$$(x, y) \notin A^n_m.$$

The complement of A^n_m is Σ^1_{n+3}. The witness to this Σ^1_{n+3} statement is preserved through the remaining tail by the T_{n+1} absoluteness supplied by Lemma 2.27. Hence $(x, y) \notin A^n_m$ in W^*_n. Since y was arbitrary, the section is empty. □

Corollary 5.8 (The M_n uniformizing relations)**.** *In W^*_n, every Π^1_{n+3} set of pairs of reals has a Π^1_{n+3} uniformization. More precisely, for each $m \in \omega$, define*

$$U^n_m(x, y) \quad \Longleftrightarrow \quad (x, y) \in A^n_m \ \wedge \ \neg\Phi_n(\langle \mathrm{UF}, n, x, y, m \rangle).$$

Then U^n_m uniformizes A^n_m.

Proof. Since A^n_m is Π^1_{n+3} and Φ_n is Σ^1_{n+3}, the displayed relation is Π^1_{n+3}. If the x-section of A^n_m is empty, then the x-section of U^n_m is empty. If the x-section of A^n_m is nonempty, Lemma 5.7 gives a unique value y_0 in the section whose UF-tag is not Φ_n-coded. Hence $U^n_m(x, y)$ holds exactly for $y = y_0$. This is uniformization. The boldface version follows by the usual coding of real parameters into the universal family and by the fact that the bookkeeping ranges over names for those parameters. □

Lemma 5.9 (The Δ^1_{n+3} well-order in the M_n extension)**.** *In W^*_n, the reals admit a Δ^1_{n+3}-definable well-order.*

Proof. For a real $z \in W_n^*$, let $\pi_n(z)$ be the least canonical M_n-localized presentation of a name whose interpretation is z, ordered by the fixed M_n-well-order of localized presentations. Every real has such a presentation because the final hybrid forcing has countable support and every real name has bounded local support. Distinct reals have distinct least presentations.

Define the underlying order by

$$z_0 \lhd_n z_1 \quad \Longleftrightarrow \quad z_0 \neq z_1 \text{ and } \pi_n(z_0) <_{M_n} \pi_n(z_1).$$

This is a well-order of the reals. The well-order stages of Definition 5.5 intentionally code

$$\langle \mathrm{WO}, n, z_0, z_1 \rangle$$

exactly when $z_0 \lhd_n z_1$. By Lemma 5.2, for distinct reals z_0, z_1,

$$z_0 \lhd_n z_1 \quad \Longleftrightarrow \quad \Phi_n(\langle \mathrm{WO}, n, z_0, z_1 \rangle).$$

Thus

$$z_0 <_{\Delta,n} z_1 \quad \Longleftrightarrow \quad z_0 \neq z_1 \ \wedge \ \Phi_n(\langle \mathrm{WO}, n, z_0, z_1 \rangle)$$

is a Σ^1_{n+3} definition of the well-order. Since exactly one of the two opposite well-order tags is intentionally coded, the same relation is equivalently given by

$$z_0 <_{\Delta,n} z_1 \quad \Longleftrightarrow \quad z_0 \neq z_1 \ \wedge \ \neg\Phi_n(\langle \mathrm{WO}, n, z_1, z_0 \rangle).$$

This is a Π^1_{n+3} definition. Hence the well-order is Δ^1_{n+3}. ☐

Lemma 5.10 (Localized covering over $L[T_{n+1}, a]$)**.** *Let $a \in W_n^*$ be a real and put*

$$N_a^n = L[T_{n+1}, a].$$

Then there is a Borel null set $Z^{\mathcal{N}}_{n,a} \in W_n^*$ such that

$$\bigcup \{B \mid B \text{ is a Borel null set coded in } N_a^n\} \subseteq Z^{\mathcal{N}}_{n,a}.$$

There is also a Borel meager set $Z^{\mathcal{M}}_{n,a} \in W_n^*$ such that

$$\bigcup \{B \mid B \text{ is a Borel meager set coded in } N_a^n\} \subseteq Z^{\mathcal{M}}_{n,a}.$$

Consequently the set of reals which are not Random-generic over N_a^n is null, and the set of reals which are not Cohen-generic over N_a^n is meager.

Proof. Fix $a \in W_n^*$. By the M_n-version of Lemma 3.10, choose a countable admissible support $A \subseteq \kappa_n$ and a $\mathbb{P}_A$-name σ such that

$$\sigma^{G_A} = a,$$

where $G_A = G^n_\infty \cap \mathbb{P}_A$. Choose $\beta < \kappa_n$ with $A \subseteq \beta$. Let $\dot{a} = \sigma^{\uparrow\beta}$, and let $\dot{N}$ be the $\mathbb{P}_\beta$-name for $L[T_{n+1}, \dot{a}]$. Then the corresponding regularity tags for

$\dot{N}$ are admissibly based. This is the higher-level version of Definition 3.6, with

$$A_{\mathrm{rb}} = A, \qquad k = 1, \qquad \dot{a}_0 = \sigma, \qquad \dot{W}_{\mathrm{rb}} = L[T_{n+1}, \sigma].$$

The set B^0_{rb} is the countable union of the branch-product supports of the W_n-parameters occurring in σ, and $\dot{B}_{\mathrm{rb}}$ is the corresponding branch footprint generated by B^0_{rb} and by the explicit coding coordinates in A.

By the final bookkeeping, at some stage $\lambda_{\mathcal{N}} > \beta$ the tag

$$(\mathrm{Am}_{\mathcal{N}}, \dot{N}^{\upharpoonright \lambda_{\mathcal{N}}})$$

is used. At this stage the forcing is

$$\mathbb{A}^{N^n_a}_{\mathcal{N}},$$

computed in the interpreted base N^n_a. Let $Z^{\mathcal{N}}_{n,a}$ be the Borel null set added by this amoeba forcing. By the definition of amoeba forcing for the null ideal,

$$B \subseteq Z^{\mathcal{N}}_{n,a}$$

for every Borel null set B coded in N^n_a. Later forcing preserves the Borel code for $Z^{\mathcal{N}}_{n,a}$ and the statement that it is null. Hence the first displayed inclusion holds in W^*_n.

The meager case is the same. The bookkeeping later uses the tag

$$(\mathrm{Am}_{\mathcal{M}}, \dot{N}^{\upharpoonright \lambda_{\mathcal{M}}})$$

for some $\lambda_{\mathcal{M}} > \beta$. The forcing $\mathbb{A}^{N^n_a}_{\mathcal{M}}$ adds a Borel meager set $Z^{\mathcal{M}}_{n,a}$ covering every Borel meager set coded in N^n_a, and this covering relation remains true in the final extension.

Finally, a real is not Random-generic over N^n_a iff it belongs to a Borel null set coded in N^n_a. Hence the non-Random reals over N^n_a are contained in $Z^{\mathcal{N}}_{n,a}$. Similarly, the non-Cohen reals over N^n_a are contained in $Z^{\mathcal{M}}_{n,a}$. This proves the lemma. □

Lemma 5.11 (Hjorth–Solovay regularity in the M_n extension)**.** *In W^*_n, every boldface $\boldsymbol{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable and has the Baire property.*

Proof. Let $A \subseteq \omega^\omega$ be $\Sigma^1_{n+2}(a)$, where $a \in W^*_n$ is a real parameter, and put $N^n_a = L[T_{n+1}, a]$. By Lemma 5.10, almost every real is Random over N^n_a and comeagerly many reals are Cohen over N^n_a.

For Lebesgue measurability, work in N^n_a with the Random algebra $\mathbb{B}^{N^n_a}_{\mathrm{rand}}$ and let $\dot{r}$ be the canonical Random real. The Boolean value of the statement "$\dot{r} \in A$" is represented, modulo null sets, by a Borel set B coded in N^n_a. If z is Random over N^n_a, the forcing theorem for $\mathbb{B}^{N^n_a}_{\mathrm{rand}}$ gives

$$z \in B \iff N^n_a[z] \models z \in A.$$

Since A is $\Sigma^1_{n+2}(a)$ and T_{n+1} is the canonical weakly homogeneous tree representing the universal Σ^1_{n+2} set, Lemma 2.27 identifies this truth with the truth of $z \in A$ in the final model. Therefore $A \triangle B$ is contained in the null set of reals which are not Random over N^n_a. Hence A is Lebesgue measurable.

For the Baire property, the same argument is carried out in N^n_a with Cohen forcing. The Boolean value of "$\dot{c} \in A$" for the canonical Cohen real is represented by a Borel set C coded in N^n_a, and for every Cohen real z over N^n_a we have

$$z \in C \quad \Longleftrightarrow \quad z \in A$$

by the same T_{n+1} absoluteness. Thus $A \triangle C$ is contained in the meager set of reals which are not Cohen-generic over N^n_a. Hence A has the Baire property. $\square$

5.1 Σ^1_{n+2}-uniformization in small M_n-generic extensions

We shall argue for Σ^1_{n+2}-uniformization now. Again, as in the case $n = 1$, this follows from a more general fact. Throughout this subsection, a small generic extension of M_n means an extension $M_n[G]$ by a set forcing in M_n of M_n-cardinality below the least Woodin cardinal, in the class of forcing extensions considered here, so that the canonical M_n-strategy and the relevant comparison arguments are preserved.

Proposition 5.12. *Let N be such a small generic extension of M_n, and let $s \in \mathbb{R}^N$. Let $M_n(s)$ be the canonical proper class s-mouse with n Woodin cardinals. Then the following hold in N.*

1. *If*

$$N \models \exists u\, \exists v\, \theta(s, u, v),$$

where θ is Π^1_{n+1}, then there are witnesses $u, v \in \mathbb{R} \cap M_n(s)$ such that

$$M_n(s) \models \theta(s, u, v).$$

2. *For reals $u, v \in M_n(s)$, Π^1_{n+1} truth is computed correctly by $M_n(s)$ and is preserved to N. Thus, for every Π^1_{n+1} formula θ,*

$$M_n(s) \models \theta(s, u, v) \quad \Longleftrightarrow \quad N \models \theta(s, u, v).$$

3. *The canonical mouse order $<_{n,s}$ on $\mathbb{R} \cap M_n(s)$ is a good $\Sigma^1_{n+2}(s)$ well-order. More explicitly, if $\theta(s, u, v)$ is Π^1_{n+1}, then the relation*

$$\mathrm{Least}^n_\theta(s, u)$$

saying that $u \in M_n(s)$ and u is the $<_{n,s}$-least real for which there is a $v \in M_n(s)$ with $M_n(s) \models \theta(s, u, v)$ is uniformly $\Sigma^1_{n+2}(s)$.

Proof. This is the relativized Steel analysis of the canonical mice $M_n(s)$, in the same form as Fact 4.10 for $n = 1$. We recall the ingredients in order to make clear that no new coding argument is being used here.

The projective description of the sound initial segments of $M_n(s)$ is the relativized version of the Π_n-iterability analysis recalled in Section 2. In the codes, the relevant s-premice are described by a $\Pi^1_{n+1}(s)$ condition: they are sound, project to ω, are n-small over the parameter s, and have the required Π_n-iterability. Steel comparison linearly orders these premice by initial segment and identifies the correctly iterable ones with the initial segments of the canonical mouse $M_n(s)$.

The usual comparison-and-capture argument then gives the first two clauses. If a small generic extension satisfies a $\Sigma^1_{n+2}(s)$ assertion, write it in the form $\exists u \exists v\, \theta(s, u, v)$ with θ Π^1_{n+1}. The tree or mouse witnessing this assertion is captured by a countable initial segment of $M_n(s)$; otherwise comparison with the canonical construction would produce the same contradiction as in Steel's proof of the projective correctness of $M_n(s)$. Conversely, once the witnesses belong to $M_n(s)$, Π^1_{n+1} correctness follows from the same comparison theorem and the preservation convention for the generic extensions considered here.

Finally, define $<_{n,s}$ by first taking the least sound initial segment of $M_n(s)$ containing the real in question, and then using the canonical well-order of that premouse. Since membership in the relevant initial segment class is $\Pi^1_{n+1}(s)$ and comparison gives initial-segment linearity, initial segments of $<_{n,s}$ are uniformly $\Sigma^1_{n+2}(s)$. Therefore least-witness assertions for Π^1_{n+1} matrices are again $\Sigma^1_{n+2}(s)$. □

Theorem 5.13. *Every small generic extension of M_n satisfies boldface $\mathbf{\Sigma}^1_{n+2}$-uniformization. In particular, W_n^* satisfies Σ^1_{n+2}-uniformization.*

Proof. Let N be a small generic extension of M_n. Work in N, and let $A \subseteq \mathbb{R}^2$ be a boldface $\mathbf{\Sigma}^1_{n+2}$ relation. Choose a real parameter a and a Π^1_{n+1} formula ψ such that

$$A(x, y) \iff \exists z\, \psi(a, x, y, z).$$

For each real x, put $s = a \oplus x$, and let $\theta(s, y, z)$ be the Π^1_{n+1} formula obtained from $\psi(a, x, y, z)$ after decoding s as $a \oplus x$. Define $U(x, y)$ to hold iff

$$\mathrm{Least}^n_\theta(a \oplus x, y).$$

By Proposition 5.12(3), U is $\Sigma^1_{n+2}(a)$, hence boldface $\mathbf{\Sigma}^1_{n+2}$.

If $U(x, y)$ holds, then for some $z \in M_n(a \oplus x)$,

$$M_n(a \oplus x) \models \psi(a, x, y, z).$$

By Proposition 5.12(2), $N \models \psi(a, x, y, z)$, and therefore $A(x, y)$ holds. Thus $U \subseteq A$.

The relation U is single-valued because $<_{n,a\oplus x}$ is a well-order and $U(x,y)$ asserts that y is the least real in $M_n(a \oplus x)$ for which a suitable z exists.

Finally suppose that the x-section of A is nonempty in N. Then

$$N \models \exists y \exists z\, \psi(a,x,y,z).$$

By Proposition 5.12(1), there are witnesses $y, z \in M_n(a \oplus x)$. Hence the set of $<_{n,a\oplus x}$-candidates is nonempty, and it has a least element y_0. By definition, $U(x,y_0)$ holds. Thus U uniformizes A. ☐

Theorem 5.14 (The main theorem, uniform form)**.** *Assume that M_n exists, where $1 \leqslant n < \omega$. There is a forcing extension W_n^* such that*

$$W_n^* \models 2^{\aleph_0} = \aleph_2,$$

every boldface $\boldsymbol{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable and has the Baire property, the Σ^1_{n+2}- and Π^1_{n+3}-uniformization properties hold, and the reals admit a Δ^1_{n+3}-definable well-order.

Proof. Let W_n^* be the final extension from Definition 5.5. The forcing has length $\kappa_n = (\omega_2)^{M_n}$, preserves ω_1, has the $\aleph_2$-chain condition by the same size and Δ-system argument used in Lemma 3.9, and every iterand has size at most $\aleph_1$ in the relevant intermediate model. Since the preparatory model satisfies CH, the final forcing has size at most κ_n, so

$$W_n^* \models 2^{\aleph_0} \leqslant \aleph_2.$$

Cofinally many ordinary Cohen coordinates add new reals, and hence W_n^* has at least κ_n many reals. Therefore

$$W_n^* \models 2^{\aleph_0} = \aleph_2.$$

Theorem 5.13 gives Σ^1_{n+2}-uniformization. Corollary 5.8 gives Π^1_{n+3}-uniformization. Lemma 5.9 gives a Δ^1_{n+3} well-order of the reals. Lemma 5.11 gives Lebesgue measurability and the Baire property for all boldface $\boldsymbol{\Sigma}^1_{n+2}$ sets of reals. This proves the theorem. ☐

References

[1] Andrés Eduardo Caicedo and Benedikt Löwe. The fourteen victoria delfino problems and their status in the year 2019, 2019. Manuscript.

[2] S. D. Friedman and R. Schindler. Universally baire sets and definable well-orderings of the reals. *The Journal of Symbolic Logic*, 68(4):1065–1081, 2003.

[3] G. Hjorth. The size of the ordinal u_2. *Journal of the London Mathematical Society, Second Series*, 52(3):417–433, 1995.

[4] S. Hoffelner. Forcing the Π^1_n-uniformization property. Submitted. Preprint available on arXiv.

[5] S. Hoffelner. Forcing the Π^1_3-reduction property and a failure of Π^1_3-uniformization. *Annals of Pure and Applied Logic*, 174(8):103292, 2023.

[6] Stefan Hoffelner and Sandra Müller. On a local variant of the twelfth delfino problem. Companion preprint.

[7] D. A. Martin and R. M. Solovay. A basis theorem forΣ^1_3 sets of reals. *Annals of Mathematics*, 89(1):138–159, 1969.

[8] D. A. Martin and J. R. Steel. The tree of a moschovakis scale is homogeneous. In *Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I*, pages 404–420, 2008.

[9] Y. Moschovakis. *Descriptive Set Theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2009.

[10] R. Schindler. Delfino problem #12. Notes on a talk in Bonn.

[11] Ralf Schindler. The core model for almost linear iterations. *Annals of Pure and Applied Logic*, 116(1–3):205–272, 2002.

[12] J. Steel. An outline of inner model theory. In *Handbook of Set Theory*. Springer.

[13] J. Steel. Projectively well-ordered inner models. *Annals of Pure and Applied Logic*, pages 77–104, 1995.

[14] John R Steel. The derived model theorem. In *Logic Colloquium 2006*, volume 32, pages 280–327. Cambridge University Press Cambridge, 2009.