

On graphs of total projective functions

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Abstract

It is well known that the graph of a total Σ_n^1 -function is Π_n^1 . We prove the consistency of the dual assertion at the third projective level: there is a model of ZFC in which the graph of every total Π_3^1 -function is Σ_3^1 . This principle is incompatible with Π_3^1 -uniformization and hence with the usual projective-determinacy picture. The construction also repairs the final step of the failure-of-uniformization argument from [6].

1 Introduction

The projective hierarchy is proper, and this prevents any general formal duality between Σ_n^1 and Π_n^1 uniformization statements. For graphs of total functions, however, one direction behaves differently. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a total function whose graph is Σ_n^1 , then the graph of f is also Π_n^1 , since

$$(x, y) \in \text{Graph}(f) \iff \forall z((x, z) \in \text{Graph}(f) \rightarrow z = y).$$

The same argument applies in the boldface classes, with the relevant real parameters carried along. There is no analogous formal argument for total Π_n^1 -functions. The purpose of this paper is to show that at the third projective level the corresponding dual statement is nevertheless consistent.

Theorem 1.1. *Assuming $\text{Con}(\text{ZFC})$, there is a model of ZFC in which every total Π_3^1 -function has a Σ_3^1 graph.*

This is not merely a formal curiosity. As shown below, the total-graph principle obtained in Theorem 1.1 is incompatible with the Π_3^1 -uniformization property. It therefore gives another example of a projective phenomenon which cannot occur under projective determinacy.

Uniformization, separation and reduction are among the classical regularity principles for definable sets of reals; see, for example, [15, 17, 13, 18]. Under

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projective determinacy, these principles are governed by the periodicity phenomena and scale machinery of the projective hierarchy; see [16, 14]. The present paper belongs to a different line of investigation: the study, by forcing, of the possible behaviour of uniformization, reduction and separation in the absence of projective determinacy. It is also connected with the older use of projective wellorderings and coding constructions to control definable sets of reals; see, for example, [1, 4, 2, 3]. In such forcing extensions the projective hierarchy can display patterns which are ruled out by the determinacy picture. Theorem 1.1 is one such pattern.

This work is part of a broader programme developed in [5, 6, 9, 8, 7, 10, 12]. The guiding theme of this programme is that forcing can produce finely calibrated projective universes in which the classical regularity principles separate from one another, combine in unexpected ways, or hold globally at levels where the determinacy intuition suggests a very different picture. The consistency of Σ_3^1 -separation was obtained in [5]; a model of Π_3^1 -reduction together with a failure of Π_3^1 -uniformization was constructed in [6]; forcing methods for Π_n^1 -uniformization were developed in [9]; and global Σ -uniformization phenomena in the presence of forcing axioms or large continuum are studied in [8, 7, 12, 11]. The present article adds another instance to this programme: it shows that total projective functions may have a definability behaviour which is not dictated by the formal duality between Σ and Π definitions and which is incompatible with the projective-determinacy pattern.

The proof is another instance of a fixed-point method for iterations of coding forcings. In these arguments one wants to iterate coding forcings long enough to meet all relevant requirements, but the correctness of a coding step depends on preservation by all future forcings which are still allowed. Thus the class of allowable continuations and the preservation requirements are defined simultaneously. This creates an apparent circularity. The way out is to organize the construction as a transfinite derivative on classes of allowable iterations and to work at the eventual fixed point. This method is central in the Π_3^1 -reduction construction of [6] and in the forcing construction for Π_n^1 -uniformization in [9]; a simpler version also appears in the forcing construction for Σ_3^1 -separation in [5]. The technique has recently been extended and refined in [11].

Here the fixed-point method is adapted to the total-graph problem. The new feature is the dichotomy between preservation witnesses and no-witness stages. If a local model has no preservation witness for a possible value of a total graph, then old candidates can be destroyed. If a genuinely preserved witness appears only later, it must be new over the earlier local model. A standard squaring argument then turns such a later witness into two independent witnesses, thereby forcing non-functionality. This is the mechanism which prevents no-witness, or guessing, stages from contributing junk to the final Σ_3^1 definition of the graph.

The construction uses the coding machinery from [6]. That paper correctly develops the forcing framework used for Π_3^1 -reduction, but its final extraction of a non-uniformizable relation relied on an overstrong assertion about partial Π_3^1 graphs. The present paper isolates the total-graph statement which is actually needed and proves it directly. This provides the promised repair of the final

step.

2 Preliminary considerations

We first record why Theorem 1.1 implies a failure of $\mathbf{\Pi}_3^1$ -uniformization.

Lemma 2.1. *In ZFC there is a total $\mathbf{\Pi}_3^1$ relation $R \subseteq \mathbb{R}^2$ with $\text{dom}(R) = \mathbb{R}$ which cannot be uniformized by any function whose graph is $\mathbf{\Sigma}_3^1$.*

Proof. Let $B \subseteq \mathbb{R}$ be a $\mathbf{\Sigma}_3^1$ set which is not $\mathbf{\Delta}_3^1$ in the boldface sense. Fix a $\mathbf{\Pi}_2^1$ relation P such that

$$x \in B \iff \exists y P(x, y).$$

By applying a recursive bijection if necessary, we may assume without loss of generality that

$$\forall x \forall y (P(x, y) \rightarrow y \neq 0).$$

Define $R \subseteq \mathbb{R}^2$ by

$$R(x, y) \iff P(x, y) \vee (x \notin B \wedge y = 0).$$

Then R has full first projection. Moreover, R is $\mathbf{\Pi}_3^1$, since projective pointclasses are closed under finite unions and P is $\mathbf{\Pi}_2^1 \subseteq \mathbf{\Pi}_3^1$, while the second disjunct is $\mathbf{\Pi}_3^1$.

Suppose for contradiction that $g: \mathbb{R} \rightarrow \mathbb{R}$ uniformizes R and that $\text{Graph}(g) \in \mathbf{\Sigma}_3^1$. Since g is total, its graph is also $\mathbf{\Pi}_3^1$, hence $\text{Graph}(g) \in \mathbf{\Delta}_3^1$. For every $x \in \mathbb{R}$, since g uniformizes R , $R(x, g(x))$ holds. If $x \notin B$, then $\neg \exists y P(x, y)$, so the second disjunct of R must hold and $g(x) = 0$. Conversely, if $x \in B$, then $P(x, g(x))$ holds; by the choice of P , this implies $g(x) \neq 0$. Thus

$$x \notin B \iff g(x) = 0.$$

Because $\text{Graph}(g) \in \mathbf{\Delta}_3^1$, the preimage $g^{-1}[\{0\}]$ is $\mathbf{\Delta}_3^1$. This implies $\mathbb{R} \setminus B \in \mathbf{\Delta}_3^1$, contradicting the choice of B . \square

Corollary 2.2. *In every model in which all total $\mathbf{\Pi}_3^1$ -functions have $\mathbf{\Sigma}_3^1$ graphs, the $\mathbf{\Pi}_3^1$ -uniformization property fails.*

Proof. If $\mathbf{\Pi}_3^1$ -uniformization held, the relation R from Lemma 2.1 would be uniformized by some function f_R such that $\text{Graph}(f_R) \in \mathbf{\Pi}_3^1$. By the assumed property, $\text{Graph}(f_R)$ would also be $\mathbf{\Sigma}_3^1$, contradicting Lemma 2.1. \square

Remark 2.3. The elementary observation used in the introduction has a corresponding dual use: whenever a universe satisfies the appropriate projective uniformization principle, a total $\mathbf{\Sigma}_n^1$ relation can be uniformized by a function whose graph is both $\mathbf{\Sigma}_n^1$ and $\mathbf{\Pi}_n^1$. We shall not use this observation below.

3 The base model and coding machinery

The construction uses the coding machinery developed in [6]. In this section we recall only the parts of that machinery which are used below, and we fix the notation for the present application. The flaw in [6] occurs only in the final extraction of a non-uniformizable relation. It does not affect the construction of the ground model, the coding forcings, the product closure of allowable forcings, or the preservation lemmas for allowable forcings. We shall use those parts as a black box, with the coding real replaced by a recursive code for the tuple relevant to a projective graph.

We work over a ground model W obtained by an ω -distributive forcing over L . Starting in L , let

$$\mathbb{Q} = \mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2$$

be the standard three-step forcing from [6]. The first iterand adds, by countable approximations, a sequence

$$\vec{S} = \langle S_\alpha \mid \alpha < \omega_1 \rangle$$

of pairwise independent Suslin trees over L . The subsequent two iterands locally code this sequence into the structure of $H(\omega_2)$. The forcing \mathbb{Q} is ω -distributive, hence adds no reals. If $G \subseteq \mathbb{Q}$ is L -generic and $W = L[G]$, then in W the sequence \vec{S} is uniquely characterized by a Σ_1 formula over $H(\omega_2)$ with parameter ω_1 . This definition is absolute to all generic extensions preserving ω_1 . Thus, throughout the later forcing construction, the phrase “the α -th coding tree” continues to have the same meaning in every relevant intermediate extension.

Because the final theorem is a boldface statement, real parameters must be included in the coding. We fix a recursive enumeration

$$\langle \varphi_m(v_0, v_1, v_2) \mid m < \omega \rangle$$

of all lightface Π_3^1 formulas in three displayed real variables. For a real parameter p , set

$$A_{m,p} = \{(x, y) \in \mathbb{R}^2 \mid \varphi_m(x, y, p)\}.$$

Thus every boldface Π_3^1 subset of the plane is some $A_{m,p}$. Given reals p, x, y and $m < \omega$, let (p, x, y, m) denote a fixed recursive real code for the quadruple.

We use the same local coding device as in [6]. The forcing which writes the real (p, x, y, m) into the Suslin-tree pattern is denoted by

$$\text{Code}(p, x, y, m) := \mathbb{P}_{(p,x,y,m)} = \mathbb{C}(\omega_1)^L * \dot{\mathbb{A}}(\dot{Y}).$$

Here $\mathbb{C}(\omega_1)^L$ is the forcing, computed in L , which adds a Cohen subset of ω_1 by countable conditions. The generic Cohen set chooses, via the fixed constructible bookkeeping of [6], an ω_1 -sequence of fresh ω -blocks of indices in \vec{S} . Inside each selected block the bits of the real (p, x, y, m) are represented by adding branches through one of two corresponding Suslin trees. The second factor $\dot{\mathbb{A}}(\dot{Y})$ is Jensen–Solovay almost-disjoint coding, relative to the fixed constructible

almost-disjoint family, and codes the reshaped set $Y \subseteq \omega_1$ into a single real. The reshaping is the local coding step from [6]: it ensures that countable transitive models containing the final coding real can reconstruct enough of the branch pattern to verify the intended code by a projective statement of the required complexity.

The precise definition of Y will not be repeated here. The relevant points are the following. First, Y is obtained uniformly from the Cohen reservoir and from the tuple (p, x, y, m) . Second, the almost-disjoint forcing $\mathbb{A}(Y)$ depends only on the set $Y \subseteq \omega_1$ and on the fixed almost-disjoint family from L ; it is therefore independent of the ambient intermediate universe once Y is present and ω_1 is preserved. Third, the resulting code is upward absolute in the intended sense: once the code has been introduced, the corresponding projective decoding statement remains true in all further extensions preserving the relevant coding structure.

We first record the one-step coding consequence. The corresponding iteration facts are collected below, after the definition of allowable forcings.

Lemma 3.1. *There is a recursive coding of quadruples $t = (p, x, y, m)$ by reals and a uniformly Σ_3^1 predicate*

$$\Phi(p, x, y, m)$$

with the following one-step property. Let M be any intermediate ω_1 -preserving extension in which the fixed sequence of coding trees is still Σ_1 -definable over $H(\omega_2)$. If $t = (p, x, y, m) \in M$ and one forces over M with a single coding forcing $\text{Code}(p, x, y, m)$, using a fresh coding block assigned to t , then the resulting extension satisfies

$$\Phi(p, x, y, m).$$

Moreover, the decoding on this block is exact: the block used by this instance of $\text{Code}(t)$ decodes the tuple t and no different tuple. The formula Φ is absolute to the ω_1 -preserving generic extensions considered below, because the sequence of coding trees remains Σ_1 -definable over $H(\omega_2)$.

Proof. This is the standard one-step decoding lemma for the coding apparatus of [6], applied to the recursive real code of the tuple (p, x, y, m) . The Cohen reservoir chooses a fresh block of the fixed independent Suslin-tree sequence, the branch pattern on that block represents the bits of the real coding t , and the Jensen–Solovay almost-disjoint forcing codes the reshaped set into a real. The reshaping makes the resulting block recognizable by a Σ_3^1 condition. Since the block is assigned to the single tuple t , the recovered branch pattern decodes exactly t . The definability of the underlying tree sequence over $H(\omega_2)$ gives the stated absoluteness of the decoding formula in all later ω_1 -preserving extensions. \square

We next recall the class of allowable forcings from [6]. In the present paper it is useful to formulate the definition so that the product/iteration hybrid is visible. At each stage the construction first adds a fresh copy of $\mathbb{C}(\omega_1)^L$ on

a new reservoir coordinate, independently of the generic object already constructed. Only after this independent reservoir coordinate has been added does the bookkeeping split into two cases: either the stage is padding and the second component is trivial, or the stage codes a real named in the current intermediate extension by Jensen–Solovay almost-disjoint coding. Thus the object handled by the bookkeeping is a real, not an arbitrary set $Y \subseteq \omega_1$; the set Y used by the almost-disjoint forcing is produced uniformly from that real and from the fresh reservoir generic.

Definition 3.2. *Let $\alpha < \omega_1$. A hybrid allowable presentation of length α over W consists of a sequence*

$$\langle \mathbb{P}_\beta \mid \beta \leq \alpha \rangle,$$

a bookkeeping function $F \in W$ with domain α , pairwise disjoint reservoir coordinates, and pairwise disjoint Suslin-tree coding blocks, satisfying the following clauses.

First, \mathbb{P}_0 is trivial. At a successor stage $\beta + 1$, let

$$\mathbb{C}_\beta = \mathbb{C}(\omega_1)^L$$

be the fresh reservoir forcing assigned to β . This is not a \mathbb{P}_β -name and is not interpreted inside the current \mathbb{P}_β -generic extension. The next stage is obtained by first taking the independent product with this fresh reservoir coordinate and then applying a second, iteration-like component:

$$\mathbb{P}_{\beta+1} = (\mathbb{P}_\beta \times \mathbb{C}_\beta) * \dot{\mathbb{B}}_\beta.$$

Here $\dot{\mathbb{B}}_\beta$ is a $(\mathbb{P}_\beta \times \mathbb{C}_\beta)$ -name determined by $F(\beta)$ as follows.

(1) Padding stage. *If $F(\beta) = \star$, then*

$$\dot{\mathbb{B}}_\beta = \check{\mathbf{1}}.$$

Thus the stage still adds the fresh reservoir generic for \mathbb{C}_β , but it performs no almost-disjoint coding.

(2) Coding stage. *If $F(\beta)$ is a code for a \mathbb{P}_β -name \dot{r}_β such that*

$$\mathbb{P}_\beta \Vdash \dot{r}_\beta \in 2^\omega,$$

then, after forcing with $\mathbb{P}_\beta \times \mathbb{C}_\beta$, the real

$$r_\beta = (\dot{r}_\beta)^{G_\beta}$$

is read from the current \mathbb{P}_β -generic object G_β , while the reservoir generic $c_\beta \subseteq \mathbb{C}_\beta$ is independent of G_β . In the extension by $G_\beta \times c_\beta$, let $Y_\beta \subseteq \omega_1$ be the standard reshaped set computed uniformly from r_β , from c_β , and from the fresh Suslin-tree block assigned to β . Then

$$\dot{\mathbb{B}}_\beta = \dot{\mathbb{A}}(\dot{Y}_\beta),$$

where $\mathbb{A}(Y_\beta)$ is the Jensen–Solovay almost-disjoint coding forcing, using the fixed constructible almost-disjoint family, which codes Y_β into a real.

At limit stages the support convention is mixed: the reservoir coordinates \mathbb{C}_β are taken with countable support, while the almost-disjoint coding components \mathbb{B}_β are taken with finite support. Freshness of reservoir coordinates and Suslin-tree coding blocks is part of the definition.

A forcing admitting such a presentation is called allowable, or 0-allowable. If \dot{r}_β is the canonical real code for a quadruple (p, x, y, m) , we write the corresponding coding stage as

$$\text{Code}(p, x, y, m).$$

This notation abbreviates the whole stage operation over \mathbb{P}_β : first adjoining the independent reservoir coordinate \mathbb{C}_β , and then performing the almost-disjoint coding determined by the reshaped set associated with the real coding (p, x, y, m) .

The definition should be read as saying that the construction iteratively codes reals

$$r_\beta = (\dot{r}_\beta)^{G_\beta}$$

whenever the bookkeeping presents such a name. The set $Y_\beta \subseteq \omega_1$ is merely the internal reshaped object which makes the code projectively recognizable. In particular, the bookkeeping does not choose an arbitrary $Y \subseteq \omega_1$; it chooses, or skips, a real to be coded, and Y_β is then produced by the fixed coding recipe.

The important point is that the fresh reservoir coordinate is present in both cases. A padding stage is

$$\mathbb{P}_\beta \mapsto (\mathbb{P}_\beta \times \mathbb{C}_\beta) * \mathbf{1},$$

whereas a coding stage is

$$\mathbb{P}_\beta \mapsto (\mathbb{P}_\beta \times \mathbb{C}_\beta) * \mathbb{A}(Y_\beta).$$

Thus the symbol $\dot{\mathbb{B}}_\beta$ denotes only the second, iteration-like component of the stage. The Cohen reservoir forcing \mathbb{C}_β is never merely a \mathbb{P}_β -name for the next iterand; it is an independent product coordinate added before the padding/coding decision is implemented. This is the product characteristic of the construction. The dependence of Y_β on the current real r_β , and the fact that later bookkeeping may depend on earlier coding reals, is the iteration characteristic.

Equivalently, for any fixed length $\delta < \omega_1$, one may first add by a countable-support product all reservoir generics

$$\left(\prod_{\beta < \delta} \mathbb{C}_\beta \right)_{\text{c.s.}}$$

and then perform the finite-support iteration of the almost-disjoint coding components, using the reservoir generic assigned to the relevant stage. This front-loaded presentation is canonically isomorphic to the hybrid presentation, provided the bookkeeping names are interpreted with the same regular supports.

It is often useful conceptually, because it shows that the Cohen part is a product of independent reservoirs, while all genuine coding decisions occur in the almost-disjoint coding part.

Remark 3.3 (Why the mixed iteration/product is harmless). The mixed presentation causes no additional preservation or definability difficulty. Indeed, any allowable forcing of length $\delta < \omega_1$ can be rearranged, up to the canonical isomorphism supplied by the chosen disjoint coordinates, as

$$\left(\prod_{\beta < \delta} \mathbb{C}_\beta \right)_{\text{c.s.}} * \left(\dot{*}_{\beta < \delta} \dot{\mathbb{B}}_\beta \right)_{\text{f.s.}},$$

where each \mathbb{C}_β is a fresh copy of $\mathbb{C}(\omega_1)^L$, and where $\dot{\mathbb{B}}_\beta$ is either the trivial forcing or the almost-disjoint coding forcing $\mathbb{A}(Y_\beta)$ computed from the real named at stage β and from the reservoir generic c_β . The first factor is σ -closed, since it is a countable-support product of $< \omega_1$ many copies of $\mathbb{C}(\omega_1)^L$. The second factor is a finite-support iteration of ccc almost-disjoint coding forcings. Hence ω_1 is preserved.

The rearrangement does not change the intended codes. Moving all reservoir coordinates to the front merely supplies the independent generics c_β in advance. The later finite-support iteration still computes the names \dot{r}_β in the same intermediate extensions and then applies the same uniform reshaping and almost-disjoint coding recipe. Thus the hybrid notation is only a bookkeeping convenience, not an additional forcing principle.

The product closure used later is also transparent in this presentation. If two allowable forcings are implemented with disjoint coding coordinates and disjoint reservoir coordinates, their product is obtained by taking the union, or an interleaving, of the two lists of stages and names to be coded. The countable-support product of all reservoir coordinates is still σ -closed, and the finite-support product/iteration of the almost-disjoint coding coordinates is still ccc. Since the decoding predicate looks only at the block assigned to the relevant code, adding further disjoint coding blocks cannot create a false code for a different tuple.

Lemma 3.4. *The following hold for allowable forcings over W :*

- (i) *If \mathbb{P} is an allowable hybrid presentation of length δ , then at every stage $\beta < \delta$ the fresh reservoir coordinate \mathbb{C}_β has size \aleph_1 , and*

$$\mathbb{P}_\beta \times \mathbb{C}_\beta \Vdash |\dot{\mathbb{B}}_\beta| \leq \aleph_1,$$

where $\dot{\mathbb{B}}_\beta$ is the second, iteration-like component of the stage.

- (ii) *Every allowable forcing over W preserves ω_1 .*
- (iii) *The product of two allowable forcings with disjoint reservoir coordinates and disjoint coding coordinates is allowable.*

- (iv) If $\langle \mathbb{P}_\alpha \mid \alpha \leq \omega_1 \rangle \in W$ is an ω_1 -length hybrid construction such that each countable initial segment is allowable over W , then $\mathbb{P}_{\omega_1} \Vdash \text{CH}$.
- (v) If a stage of an allowable construction explicitly performs the coding operation $\text{Code}(p, x, y, m)$, using a fresh reservoir coordinate and a fresh coding block, then the next intermediate extension satisfies $\Phi(p, x, y, m)$.
- (vi) In any allowable construction satisfying the disjoint-coordinate convention, forcing on coding coordinates disjoint from a block assigned to a tuple t cannot create or alter the decoding of that block. Equivalently, apart from codes already present in the initial model, the tuples decoded in the final extension are exactly the tuples explicitly written by stages of the construction.
- (vii) Once $\Phi(t)$ holds at some stage, it remains true in all later allowable extensions which use fresh coding coordinates. We call this persistence in what follows.
- (viii) In any countable or ω_1 -length construction satisfying the disjoint-coordinate convention and having allowable countable initial segments, if $\Phi(t)$ holds in the final model and was not already true in the initial model, then there is a least stage at which $\text{Code}(t)$ was explicitly introduced. In the base model W no future coding block already decodes a tuple, so every final instance of $\Phi(t)$ has such a first stage.

Proof. For (i), the independent reservoir coordinate is a fresh copy of $\mathbb{C}(\omega_1)^L$, hence has size \aleph_1 . After the product with this coordinate has been taken, the second component is either trivial or is Jensen–Solovay almost-disjoint coding with respect to the fixed almost-disjoint family of size \aleph_1 . Hence the displayed size bound holds in the relevant intermediate model.

For (ii), use the rearrangement from Remark 3.3. The reservoir part is a countable-support product of $< \omega_1$ many copies of $\mathbb{C}(\omega_1)^L$, hence is σ -closed. After forcing with it, the almost-disjoint coding part is a finite-support iteration of ccc forcings of length $< \omega_1$, hence is ccc. Therefore the whole allowable forcing preserves ω_1 .

For (iii), choose disjoint reservoir and coding coordinates for the two forcings. Interleave their bookkeeping functions. The rearranged form of the product is again a countable-support product of independent $\mathbb{C}(\omega_1)^L$ reservoirs followed by a finite-support iteration of almost-disjoint coding forcings, with trivial second components at padding stages. Since the coordinates are disjoint and the definition of $\mathbb{A}(Y)$ is absolute once Y is present, this interleaving is an allowable forcing and is canonically isomorphic to the product.

For (iv), every countable initial segment of the ω_1 -length hybrid construction preserves ω_1 by (ii). The usual argument from [6] then applies: each real in the final extension is added by some countable initial segment, and each such initial segment has size at most \aleph_1 and preserves ω_1 . Since the ground model satisfies CH and no countable initial segment produces more than \aleph_1 many reals, the final model still satisfies CH.

Clauses (v)–(viii) are the code-management consequences of Lemma 3.1 together with the disjoint-coordinate convention for allowable constructions. Clause (v) is exactly the one-step coding lemma applied in the intermediate model at the given stage, after the independent reservoir coordinate for that stage has been added. For (vi), the projective decoding formula reads only the branch pattern and the almost-disjoint code on the block assigned to the tuple under consideration. Later forcing on disjoint blocks may add further codes, but it does not change the pattern on the old block and cannot make an unused disjoint block decode a prescribed tuple. This gives non-interference. Clause (vii) follows from the same locality of the decoding: once the relevant block has been written and the almost-disjoint coding real has been added, later allowable forcing uses fresh blocks and preserves ω_1 , so the Σ_1 definition of the coding-tree sequence and the decoded pattern remain unchanged. Finally, for (viii), well-order the stages of the construction. If $\Phi(t)$ first becomes true at a positive stage, then by non-interference that stage must be an explicit use of $\text{Code}(t)$; limit stages add no new coding action except through their earlier stages. The preliminary model W contains no branches through the Suslin-tree coding blocks reserved for the later construction and hence no already-decoded future tuple. Therefore every final instance of $\Phi(t)$ in the constructions considered here has a least explicit introduction stage. \square

We shall repeatedly use the following consequence without further comment. Allowable forcings may be concatenated or interleaved, provided fresh coding coordinates and independent Cohen reservoirs are used. Such rearrangements do not change which tuples are decoded by Φ , do not affect the projective complexity of the decoding predicate, and preserve ω_1 . This is the precise sense in which the later construction may freely alternate between iteration language and product language.

4 The derivative operator and ∞ -allowable forcings

We next isolate the fixed point of the preservation operation. Since the argument constantly passes to subextensions of the mixed coding constructions, we first fix the support terminology used to name those subextensions. The point of the terminology is only locality: the same derivative construction will be carried out not only over the initial model W , but also over intermediate models obtained by restricting a mixed construction to an appropriate set of coordinates.

Definition 4.1 (Regular local supports). *Let M be a transitive model, with the same ordinals as W , over which the coding machinery of Section 3 is being applied, and let \mathbb{P}_β be one of the mixed coding constructions over M . If $A \subseteq \beta$, let \mathbb{P}_A denote the inherited subconstruction using the stages in A ; in the hybrid presentation this means that at each retained stage we keep the independent $\mathbb{C}(\omega_1)^L$ reservoir coordinate and, when present, the subsequent almost-disjoint*

coding component, with all names restricted to the retained coordinates. We call A a regular support for \mathbb{P}_β if this inherited subconstruction is a regular subforcing,

$$\mathbb{P}_A \leq \mathbb{P}_\beta.$$

If G_β is \mathbb{P}_β -generic over M , then $G_A = G_\beta \cap \mathbb{P}_A$ is \mathbb{P}_A -generic over M , and $M[G_A]$ is the local intermediate model in which names supported by A are evaluated. Initial segments are included as the special case $A = \alpha < \beta$.

A bookkeeping function for a presentation of an allowable forcing may mention regular supports and nice names supported by them. We do not require arbitrary presentations used below to be exhaustive; this convention is useful for closure under products, quotients, and cones. A bookkeeping function is called exhaustive if it lists, cofinally often, every tuple consisting of a countable regular support and nice names for the relevant reals and integer parameters. The final ω_1 -length construction in Section 5 will use exhaustive bookkeeping.

With this terminology in place, the derivative is formulated locally. If M is one of the local intermediate models just described, the same recursive definition can be carried out over M using fresh reservoir and coding coordinates. We write

$$\Gamma_\alpha^M$$

for the class obtained over M at stage α of the derivative. When $M = W$ the superscript is omitted. Thus the notation Γ_∞^M below means the eventual stable class of the hierarchy computed over M ; it is not important that the first stabilizing ordinal be the same for all local models. All definitions are uniform in a code for the local presentation.

The role of the derivative is as follows. At a local stage the bookkeeping presents reals p, x, z and an index m . If there is a value y such that

$$\varphi_m(x, y, p)$$

is stable under all further forcings in the current derivative class, then y is a preservation witness and the construction may safely code (p, x, y, m) . If no such value exists, the derivative hierarchy does not attempt to decide the section. It merely codes the bookkeeping guess z . In the final witnessing construction, a no-witness stage will additionally destroy the guessed value; if a new preservation witness later appears, the old no-witness stage is used only as a marker for the squaring argument.

The point to keep in mind is that a guessing code is never treated as evidence that the guessed real belongs to $A_{m,p}(x)$. It records only that the local preservation derivative was empty.

Definition 4.2 (Local preservation witnesses). *Assume that Γ_δ^M has already been defined. For reals $p, x \in M$ and $m < \omega$, put*

$$Y_{M,x,p}^\delta = \{y \in M \cap \mathbb{R} \mid M \models \varphi_m(x, y, p) \text{ and } \forall \mathbb{R} \in \Gamma_\delta^M \mathbb{R} \Vdash_M \varphi_m(\check{x}, \check{y}, \check{p})\}.$$

The elements of $Y_{M,x,p,m}^\delta$ are the local δ -preservation witnesses for (m, p, x) over M . If $M = W[G_A]$, we also write $Y_{A,x,p,m}^\delta$.

If a nonempty preservation set has to be represented by one of its elements, we choose the representative by the fixed well-order of names and conditions in the ambient ground model. This is only a definability convention; no uniqueness of preservation witnesses is assumed.

In a presentation of a forcing in the derivative hierarchy we keep three auxiliary records. The record I contains preservation entries of the form

$$(A, p, x, y, m, \rho),$$

meaning that, over the local model determined by the regular support A , the real y was chosen as a local ρ -preservation witness for (m, p, x) . The record J contains guessing entries of the same form, but with y replaced by the bookkeeping guess z ; such an entry asserts only that the relevant preservation set was empty at that local stage. Finally, D contains certificates of non-functionality, for instance two distinct recorded preservation witnesses for the same triple (m, p, x) . The records are part of the presentation and are used only in the verification.

Definition 4.3 (Successor step). *Assume that the classes Γ_δ^N have been defined uniformly for all local models N arising from regular supports. A forcing belongs to $\Gamma_{\delta+1}^M$ if it belongs to Γ_δ^M and admits a presentation as a mixed allowable iteration, with records I, J, D , satisfying the following rule at each nontrivial stage.*

The bookkeeping at the stage presents a regular support A and nice \mathbb{P}_A -names for reals p, x, z and for an integer m . After evaluating these names in the local model $N = M[G_A]$, compute

$$Y_{N,x,p,m}^\delta$$

in N . The stage first supplies an independent Cohen reservoir for fresh coding coordinates. Then exactly one of the following alternatives is used.

- (a) **Local preservation.** *If $Y_{N,x,p,m}^\delta \neq \emptyset$, let y be the representative chosen by the fixed well-order and force with*

$$\text{Code}(p, x, y, m).$$

Add (A, p, x, y, m, δ) to I . If either $Y_{N,x,p,m}^\delta$ contains two distinct reals, or a distinct preservation witness for the same triple (m, p, x) has already been recorded in I , add the corresponding certificate to D .

- (b) **Pure guessing.** *If $Y_{N,x,p,m}^\delta = \emptyset$, force only with the coding forcing for the bookkeeping guess,*

$$\text{Code}(p, x, z, m).$$

Add the corresponding guessing entry to J . No destruction forcing is used in the derivative hierarchy, and no element is added to I in this case.

At limit stages of an iteration we take the direct limit of the forcing and the unions of the records.

For a limit ordinal λ , define, locally over each M ,

$$\Gamma_\lambda^M = \bigcap_{\alpha < \lambda} \Gamma_\alpha^M.$$

Lemma 4.4. *For every ordinal α , the relation $\mathbb{P} \in \Gamma_\alpha^M$ is definable over the local model M , uniformly in M and in the presentation.*

Proof. The proof is by induction on α . At level 0 this is the definability of the original allowable coding iterations. At a successor level, membership in $\Gamma_{\delta+1}^M$ requires membership in Γ_δ^M together with the local rule from Definition 4.3. The latter can be expressed by quantifying over presentations, regular supports, nice names, and the records I, J, D . The definition of $Y_{N,x,p,m}^\delta$ quantifies over Γ_δ^N as computed in the local model $N = M[G_A]$, and this is available by the induction hypothesis. At limit levels definability follows by intersection. \square

Lemma 4.5. *If $\alpha < \beta$, then $\Gamma_\beta^M \subseteq \Gamma_\alpha^M$ for every local model M . Equivalently, every β -allowable forcing over M is α -allowable over M .*

Proof. This is immediate from the definition at successor stages and from the intersection definition at limit stages. \square

Lemma 4.6. *For every local model M and every ordinal α , the class Γ_α^M is closed under regular subiterations, quotients over regular supports, cones below conditions, countable concatenations, and countable mixed-support products, provided fresh coding coordinates and independent Cohen reservoirs are used. In particular:*

- (i) *if $\mathbb{P} \in \Gamma_\alpha^M$ and $q \in \mathbb{P}$, then the cone $\mathbb{P} \upharpoonright q$ belongs to Γ_α^M , after the harmless renumbering of coordinates;*
- (ii) *if $\langle \mathbb{P}^n : n < \omega \rangle$ is a sequence of forcings in Γ_α^M , then*

$$\prod_{n < \omega}^{\text{mix}} \mathbb{P}^n \in \Gamma_\alpha^M;$$

- (iii) *if $\mathbb{P} \in \Gamma_\alpha^M$ and*

$$\mathbb{P} \Vdash \dot{\mathbb{R}} \in \Gamma_\alpha^{M[\dot{G}]},$$

*then $\mathbb{P} * \dot{\mathbb{R}}$ is again α -allowable over M , after assigning fresh coordinates to the second factor.*

Proof. Let $C_\alpha(M)$ denote the assertion of the lemma for the local model M . We prove $C_\alpha(M)$ by induction on α , uniformly in M . For $\alpha = 0$ this is the closure property of the basic hybrid construction, recorded in Lemma 3.4 and the paragraph following it.

Assume $C_\delta(N)$ for all local N , and put $\alpha = \delta + 1$. For $a = (m, p, x)$ write

$$Y_N^\delta(a) = Y_{N,x,p,m}^\delta.$$

In each case below the induction hypothesis gives membership in Γ_δ^M ; the only point is that the successor rule from Definition 4.3 is preserved.

Regular subiterations and quotients are immediate from the same presentation. If A is a regular support for a presentation \mathcal{E} of \mathbb{P} , then $\mathcal{E} \upharpoonright A$ and the quotient presentation \mathcal{E}/A have the same local tests $Y_N^\delta(a)$, computed in the corresponding regular local models. Hence the preservation/guessing decision is unchanged.

(i) *Cones.* Let $q \in \mathbb{P}$. Close $\text{supp}(q)$ to a countable regular support and put $\mathbb{P}_q = \mathbb{P} \upharpoonright q$. The presentation of \mathbb{P}_q is the regular quotient presentation below q . At every stage it computes the same $Y_N^\delta(a)$ as the original presentation, with the extra condition $q \in G$. Thus the same successor decision is made, and $\mathbb{P}_q \in \Gamma_{\delta+1}^M$.

(ii) *Products.* Let

$$\mathbb{Q} = \prod_{n < \omega}^{\text{mix}} \mathbb{P}^n, \quad \mathbb{P}^n \in \Gamma_{\delta+1}^M.$$

Choose presentations \mathcal{E}_n and disjoint shift maps π_n for stages, reservoirs and coding coordinates. The product presentation is $\mathcal{E} = \bigoplus_n \pi_n(\mathcal{E}_n)$, with

$$I^\mathbb{Q} = \bigcup_n \pi_n \text{“} I^{\mathbb{P}^n} \text{”}, \quad J^\mathbb{Q} = \bigcup_n \pi_n \text{“} J^{\mathbb{P}^n} \text{”}.$$

The record $D^\mathbb{Q}$ contains the shifted old certificates, together with the new certificates coming from two shifted preservation records for the same triple (m, p, x) . If a shifted stage from the n -th factor is checked over $N' = \pi_n(N)$, then, under the canonical shift isomorphism,

$$Y_{N'}^\delta(\pi_n(a)) = \pi_n \text{“} Y_N^\delta(a) \text{”}.$$

Thus the shifted stage makes exactly the same local decision as the original stage, and $\mathbb{Q} \in \Gamma_{\delta+1}^M$.

(iii) *Concatenations.* Suppose $\mathbb{P} \Vdash \dot{\mathbb{R}} \in \Gamma_{\delta+1}^{M[G]}$. Concatenate a presentation $\mathcal{E}_\mathbb{P}$ of \mathbb{P} with a \mathbb{P} -name $\dot{\mathcal{E}}_{\dot{\mathbb{R}}}$ for a presentation of $\dot{\mathbb{R}}$, using fresh coordinates in the second block. Stages in the first block satisfy the successor rule over M ; stages in the second block satisfy it over $M[G]$. Hence $\mathbb{P} * \dot{\mathbb{R}} \in \Gamma_{\delta+1}^M$. Countable concatenations follow by iterating this argument.

At a limit λ , apply the induction result to every $\xi < \lambda$ and intersect the classes Γ_ξ^M . \square

Lemma 4.7. *For every local model M , the decreasing sequence $\langle \Gamma_\alpha^M \mid \alpha \in \text{Ord} \rangle$ stabilizes. The stabilization is uniform in the sense that the definition of the stable class is obtained by the same recursion in every local model.*

Proof. In a fixed local model M , all forcings under consideration are coded by members of a fixed set of M : their hereditary size is at most \aleph_1^M , and their presentations use only the fixed coding apparatus and subsets of ω_1^M . Thus $\langle \Gamma_\alpha^M \mid \alpha \in \text{Ord} \rangle$ is a decreasing sequence of subsets of a set. It cannot be strictly decreasing through all ordinals. The recursion defining the sequence is the same in all local models, which gives the claimed uniformity. \square

Definition 4.8. For a local model M , let Γ_∞^M denote the stable value of the local hierarchy over M . A forcing is called ∞ -allowable over M if it belongs to Γ_∞^M . If $M = W$, we simply say ∞ -allowable.

Lemma 4.9 (Fixed-point dichotomy). *Let M be a local model and let $p, x \in M \cap \mathbb{R}$ and $m < \omega$. Exactly one of the following alternatives holds in M .*

- (i) **Preservation.** *There is a real $y \in M$ such that every $\mathbb{R} \in \Gamma_\infty^M$ forces $\varphi_m(\check{x}, \check{y}, \check{p})$.*
- (ii) **Pointwise destruction.** *For every real $z \in M$, either $M \models \neg\varphi_m(x, z, p)$, or there is a forcing $\mathbb{R}_z \in \Gamma_\infty^M$ such that*

$$\mathbb{R}_z \Vdash_M \neg\varphi_m(\check{x}, \check{z}, \check{p}).$$

Proof. The first alternative says precisely that $Y_{M,x,p,m}^\infty \neq \emptyset$. If it fails, fix $z \in M$. If $M \models \neg\varphi_m(x, z, p)$, there is nothing to prove. Otherwise $M \models \varphi_m(x, z, p)$, but z is not preserved by all ∞ -allowable continuations. Hence there is $\mathbb{P} \in \Gamma_\infty^M$ such that $\mathbf{1}_\mathbb{P}$ does not force $\varphi_m(\check{x}, \check{z}, \check{p})$. Choose a condition $q \in \mathbb{P}$ with

$$q \Vdash_\mathbb{P} \neg\varphi_m(\check{x}, \check{z}, \check{p}).$$

By the cone-closure clause of Lemma 4.6, $\mathbb{P} \upharpoonright q$ belongs to Γ_∞^M and forces the displayed negation. The two alternatives are mutually exclusive by definition. \square

We shall use the following standard absoluteness observation without further comment. If $M \subseteq N$ are transitive models of ZFC with the same ordinals, N is a forcing extension of M , and $a \in M$ is a real parameter, then every $\Sigma_3^1(a)$ statement true in M remains true in N ; equivalently, every $\Pi_3^1(a)$ statement true in N was already true in M . This is the usual consequence of Shoenfield absoluteness, after writing a Σ_3^1 statement as $\exists u \psi(u, a)$ with ψ of complexity Π_2^1 .

Lemma 4.10. *Let \mathbb{P}_θ be an ∞ -allowable iteration over a local model, with the records described above, and let G_θ be generic. In the corresponding extension the following hold.*

- (1) *If $(A, p, x, y, m, \rho) \in I_\theta$, where ρ is either an ordinal stage or the stable symbol ∞ , then*

$$\varphi_m(x, y, p).$$

- (2) If D_θ contains a certificate consisting of two distinct recorded preservation witnesses for the same triple (m, p, x) , then $A_{m,p}$ is not the graph of a function.
- (3) Consequently, if $A_{m,p}$ is the graph of a function in the extension, then for each x there is at most one y occurring in an entry $(A, p, x, y, m, \rho) \in I_\theta$.

Proof. For (1), suppose that the entry was added at some stage over the local model N . There y was chosen so that every Γ_ρ^N -allowable continuation forces $\varphi_m(x, y, p)$; for $\rho = \infty$ this means membership in the stable class Γ_∞^N . By Lemma 4.6, the quotient from N to any later regular support, and in particular to the ambient extension under consideration, is such a continuation. Hence $\varphi_m(x, y, p)$ holds there.

For (2), apply (1) to the two distinct preservation witnesses. They give two distinct elements of the same vertical section $A_{m,p}(x)$. Clause (3) is the contrapositive of this observation together with the definition of the record D . \square

Lemma 4.11. *Let \mathbb{P}_{ω_1} be an ω_1 -length construction such that every countable initial segment is ∞ -allowable, and let G_{ω_1} be generic. Let I and D be the unions of the corresponding records.*

- (1) If $(A, p, x, y, m, \rho) \in I$, then

$$W[G_{\omega_1}] \models \varphi_m(x, y, p).$$

- (2) If D contains a certificate consisting of two distinct preservation witnesses for the same (m, p, x) , then $A_{m,p}$ is not the graph of a function in $W[G_{\omega_1}]$.

Proof. For (1), let the preservation witness be recorded at a countable stage $\alpha < \omega_1$. By Lemma 4.10, every countable later initial segment $W[G_\beta]$, $\beta > \alpha$, satisfies $\varphi_m(x, y, p)$. If the final model satisfied $\neg\varphi_m(x, y, p)$, then this Σ_3^1 statement would have a real witness u in the final model. Every real in the final extension appears in some countable initial segment, so choose $\beta > \alpha$ with $u \in W[G_\beta]$. Writing $\neg\varphi_m$ as $\exists u \psi(u, x, y, p)$ with ψ of complexity Π_2^1 , Shoenfield absoluteness would make the same witness u verify $\neg\varphi_m(x, y, p)$ in $W[G_\beta]$, a contradiction. Clause (2) follows by applying (1) to the two recorded witnesses. \square

5 The witnessing universe

We now run the final construction. This is an ω_1 -length construction over W . It is not, literally, an element of Γ_∞^W , since the members of Γ_∞^W have countable length. Rather, every countable initial segment is ∞ -allowable. This is the formulation needed for the preservation and cardinal arguments recalled in Section 3.

The final construction uses exhaustive bookkeeping in the sense of Definition 4.1. Thus every countable regular support, together with nice names for p, x, z and m , is considered cofinally often. At each stage fresh coding coordinates and an independent Cohen reservoir are used. The construction maintains records I, J, D as above. A code introduced from a preservation action is recorded in I ; a code introduced from a no-witness stage is recorded in J .

Suppose stage β presents a regular support $A \subseteq \beta$ and local data p, x, z, m evaluated in $M = W[G_A]$. Compute $Y_{M,x,p,m}^\infty$ in M .

- (A) **Preservation action.** If $Y_{M,x,p,m}^\infty \neq \emptyset$, choose the representative $y \in Y_{M,x,p,m}^\infty$. Add (A, p, x, y, m, ∞) to I and force with $\text{Code}(p, x, y, m)$ unless the same code has already been introduced.

If a distinct preservation witness for the same triple (m, p, x) has already been recorded, add the corresponding certificate to D . If a guessing entry (B, p, x, w, m, ∞) for the same triple has already been recorded, then the following distinction is made. If $w = y$, the earlier guessing-destruction action has already made $\neg\varphi_m(x, w, p)$ persistent, so this case cannot occur with $y \in Y_{M,x,p,m}^\infty$. If $w \neq y$, append the squaring continuation supplied by Lemma 5.5 below and record the corresponding non-functionality certificate in D .

- (B) **Guessing-destruction action.** If $Y_{M,x,p,m}^\infty = \emptyset$, first introduce the guessing code

$$\text{Code}(p, x, z, m)$$

and add the corresponding entry (A, p, x, z, m, ∞) to J . Then use the fixed-point dichotomy. If $M \models \neg\varphi_m(x, z, p)$, append the trivial forcing. Otherwise choose, by the fixed well-order, an ∞ -allowable forcing $\mathbb{R}_z \in \Gamma_\infty^M$ such that

$$\mathbb{R}_z \Vdash_M \neg\varphi_m(\check{x}, \check{z}, \check{p}),$$

and append this local destruction forcing with fresh coordinates.

Lemma 5.1. *Every countable initial segment of the final construction is ∞ -allowable.*

Proof. The proof is by induction on the length of the initial segment. At a preservation stage we use a coding step allowed by the stable local rule. At a no-witness stage the initial coding is the pure guessing step allowed by Definition 4.3 at the stabilized level, and the subsequently appended destruction forcing belongs to Γ_∞^M over the relevant local model M . At a squaring stage the forcing is the quotient of a product of two ∞ -allowable copies of the relevant tail, hence is ∞ -allowable by Lemma 4.6. Countable limits and countable concatenations are handled by Lemma 4.6. \square

Lemma 5.2. *Let M be a local model and suppose that $Y_{M,x,p,m}^\infty = \emptyset$ in M . Let $\mathbb{T} \in \Gamma_\infty^M$ and let H be \mathbb{T} -generic over M . Then no real $r \in M$ is an ∞ -preservation witness for (m, p, x) in $M[H]$.*

Proof. Fix $r \in M$. If $M \models \neg\varphi_m(x, r, p)$, then $M[H] \models \neg\varphi_m(x, r, p)$ by the Σ_3^1 -upward absoluteness observation above, since the negation of φ_m is Σ_3^1 . Thus r is not a preservation witness in $M[H]$.

Assume instead that $M \models \varphi_m(x, r, p)$. By Lemma 4.9, choose $\mathbb{R}_r \in \Gamma_\infty^M$ forcing $\neg\varphi_m(\check{x}, \check{r}, \check{p})$ over M . By Lemma 4.6, the product/concatenation of \mathbb{T} with a fresh copy of \mathbb{R}_r is ∞ -allowable over M , and after forcing with \mathbb{T} the copied \mathbb{R}_r is an ∞ -allowable continuation over $M[H]$. In the \mathbb{R}_r -extension of M , the statement $\neg\varphi_m(x, r, p)$ is true; since this statement is Σ_3^1 , it remains true after adjoining the independent \mathbb{T} -generic by the Σ_3^1 -upward absoluteness observation above. Therefore the copied continuation over $M[H]$ destroys $\varphi_m(x, r, p)$, and r is not protected in $M[H]$. \square

Lemma 5.3. *Let M be a local model satisfying CH, and suppose that $Y_{M,x,p,m}^\infty = \emptyset$ in M . There is an ω_1 -length construction over M , all of whose countable initial segments are ∞ -allowable, which forces*

$$\forall z \in M \cap \mathbb{R} \neg\varphi_m(x, z, p).$$

Proof. Enumerate $M \cap \mathbb{R}$ as $\langle z_\xi : \xi < \omega_1 \rangle$. At stage ξ , if $\neg\varphi_m(x, z_\xi, p)$ already holds, do nothing. Otherwise use the pointwise destruction forcing for z_ξ given by Lemma 4.9. Every countable initial segment is a countable concatenation of ∞ -allowable forcings, hence is ∞ -allowable by Lemma 4.6. Once $\neg\varphi_m(x, z_\xi, p)$ has been made true, it persists through the remaining forcing by the Σ_3^1 -upward absoluteness observation above. \square

We shall also use the standard product-name separation principle for mutually generic copies. Let M be transitive, let $\mathbb{T} \in M$ be a forcing, and let $\dot{r} \in M$ be a \mathbb{T} -name for a real. If \mathbb{T}_0 and \mathbb{T}_1 are coordinate-disjoint copies of \mathbb{T} , with copied names \dot{r}_0 and \dot{r}_1 , then any product condition forcing $\dot{r}_0 = \dot{r}_1$ already forces both interpretations to be the same real of M . Consequently, if $H_0 \subseteq \mathbb{T}_0$ is M -generic and $\dot{r}_0^{H_0} \notin M$, then over $M[H_0]$ the fresh copy \mathbb{T}_1 densely forces $\dot{r}_1 \neq \dot{r}_0^{H_0}$. This is the familiar separation principle for names in products of mutually generic copies, used throughout Solovay-style forcing arguments; cf. [?].

Lemma 5.4. *Let M be a local model and let $\mathbb{T} \in \Gamma_\infty^M$ be a regular tail of one of the presentations used in the final construction. Let \mathbb{T}^* be a coordinate-disjoint copy of \mathbb{T} , with the copied bookkeeping function, regular supports, names, and records. Then the product $\mathbb{T} \times \mathbb{T}^*$ is ∞ -allowable over M , and the coordinate shift sends each preservation or guessing decision in the first tail to the corresponding decision in the copied tail. In particular, if a stage of \mathbb{T} records a preservation witness named by \dot{y} for (m, p, x) , then the copied stage of \mathbb{T}^* records the copied witness \dot{y}^* for the same (m, p, x) .*

Proof. The product is ∞ -allowable by Lemma 4.6. The rest is an induction along the presentation of the tail. At each stage, the coordinate shift identifies the local support, the evaluated names, and the local model in the copied tail with their originals. Since the definition of the preservation sets Y^∞ is uniform in the local

model and the copied tail uses fresh coding coordinates, the preservation-or-guessing decision is invariant under the shift. The records I, J, D are transported by the same shift map. \square

Lemma 5.5. *Suppose that, at some earlier stage, the final construction records a guessing entry for (m, p, x) over a local model M with $Y_{M,x,p,m}^\infty = \emptyset$. Suppose further that, after an ∞ -allowable regular tail from M , a later preservation action for the same (m, p, x) chooses a witness y . If the earlier guessed value is distinct from y , then there is an ∞ -allowable continuation over the later stage which forces that $A_{m,p}$ is not the graph of a function.*

Proof. Let \mathbb{T} be the quotient, over the earlier local model M , of the regular tail leading from the no-witness stage to the later preservation action. Thus the later local model has the form $M[H_0]$ for an M -generic $H_0 \subseteq \mathbb{T}$, after the usual identification of regular supports. Let \dot{y} be the \mathbb{T} -name whose interpretation by H_0 is the chosen preservation witness $y_0 = y$.

By Lemma 5.2, no real of M can become an ∞ -preservation witness for (m, p, x) after the tail. Since y_0 is such a witness in $M[H_0]$, we have $y_0 \notin M$.

Work over the later stage, which contains H_0 and y_0 . Take a fresh copy \mathbb{T}_1 of the same tail, using disjoint coding coordinates and an independent Cohen reservoir. By Lemma 5.4, the product of the original tail and the copied tail is ∞ -allowable, and over $M[H_0]$ the copied tail is an ∞ -allowable continuation. Let \dot{y}_1 be the copied name. By the product-name separation principle stated above, the fresh copy forces

$$\dot{y}_1 \neq \check{y}_0.$$

The copied tail replays the bookkeeping from the earlier no-witness stage to the later preservation stage. Therefore it records $y_1 = \dot{y}_1$ as a preservation witness for the same triple (m, p, x) . The original branch has already recorded y_0 . By Lemma 4.10, both

$$\varphi_m(x, y_0, p) \quad \text{and} \quad \varphi_m(x, y_1, p)$$

hold after the product continuation, and the continuation forces $y_0 \neq y_1$. The corresponding two-witness certificate is placed in D . By Lemma 4.11, this non-functionality persists through the rest of the final ω_1 -length construction. \square

Remark 5.6 (The role of guessing codes). The previous lemma is the only point at which guessing codes are used. The earlier code $\Phi(p, x, w, m)$ is not used as a second member of the section $A_{m,p}(x)$. Its role is to mark a local model M in which the preservation set for (m, p, x) was empty. Therefore old reals from M cannot later become protected. If a preservation witness appears later, it is genuinely new over M , and two independent copies of the tail produce two distinct protected witnesses.

Let G_{ω_1} be generic for the final ω_1 -length construction, and write

$$W^* = W[G_{\omega_1}].$$

Every real of W^* is represented by a name with countable support; after closing this support under the finitely many dependencies relevant to the name, we may regard it as belonging to some countable regular support. By exhaustive bookkeeping, every such local datum is considered cofinally often.

Lemma 5.7. *In W^* , suppose that $A_{m,p}$ is the graph of a total function. Then for all reals x, y ,*

$$W^* \models \Phi(p, x, y, m) \iff W^* \models \varphi_m(x, y, p).$$

Consequently $A_{m,p}$ is $\Sigma_3^1(p)$, defined by

$$(x, y) \in A_{m,p} \iff \Phi(p, x, y, m).$$

Proof. Assume that $A_{m,p}$ is the graph of a total function in W^* , and denote this function by f .

We first prove that no guessing entry for this fixed pair (m, p) can occur on a section which is total in the final model. Suppose, toward a contradiction, that a guessing entry (A, p, x, w, m, ∞) was recorded over an earlier local model $M = W[G_A]$ with $Y_{M,x,p,m}^\infty = \emptyset$. Let $y = f(x)$.

There are two cases. If $w = y$, then the guessing-destruction action at the earlier stage either already saw $M \models \neg\varphi_m(x, w, p)$ or appended an ∞ -allowable forcing making this negation true. Since $\neg\varphi_m$ is Σ_3^1 , the Σ_3^1 -upward absoluteness observation above makes the negation persistent to the final model. This contradicts $w = y = f(x)$.

Assume next that $w \neq y$. Choose a countable regular support B containing A and supporting names for p, x, y . Since $W^* \models \varphi_m(x, y, p)$ and φ_m is Π_3^1 , the downward direction of the Π_3^1 -downward form of the absoluteness observation above gives $W[G_B] \models \varphi_m(x, y, p)$. By exhaustive bookkeeping, a later stage presents the local datum (B, p, x, y, m) .

At that later stage the no-witness case cannot apply. If it did, the construction would append an ∞ -allowable forcing which makes $\neg\varphi_m(x, y, p)$ true, and this Σ_3^1 statement would persist to W^* by the Σ_3^1 -upward absoluteness observation above, contradicting $y = f(x)$. Hence the preservation case applies. Let y_0 be the preservation witness chosen at that stage. Lemma 4.11 gives $W^* \models \varphi_m(x, y_0, p)$. Since $A_{m,p}$ is the graph of the function f and $W^* \models \varphi_m(x, y, p)$, we have $y_0 = y$. The earlier guessing value w is therefore distinct from the later preservation witness y_0 , and Lemma 5.5 supplies a continuation which forces two distinct members of $A_{m,p}(x)$. By the construction and Lemma 4.11, this non-functionality persists to W^* , a contradiction. Thus no such guessing entry exists.

Now suppose $W^* \models \Phi(p, x, w, m)$. By Lemma 5.1, every countable initial segment of the final construction is ∞ -allowable and hence allowable. Thus the first-stage clause of Lemma 3.4 applies, and the code has a first stage at which it was explicitly introduced. It cannot have been introduced by a guessing-destruction action, by the preceding paragraph. Therefore it was introduced by

a preservation action, and the corresponding entry belongs to I . Lemma 4.11 gives

$$W^* \models \varphi_m(x, w, p).$$

This proves the forward implication.

Conversely, suppose $W^* \models \varphi_m(x, y, p)$. Since $A_{m,p}$ is a graph, we have $y = f(x)$. Choose a countable regular support B with $p, x, y \in W[G_B]$. Again, downward Π_3^1 absoluteness gives $W[G_B] \models \varphi_m(x, y, p)$. By exhaustive book-keeping, some later stage presents (B, p, x, y, m) .

The no-witness case cannot apply at that stage, for otherwise the construction would destroy $\varphi_m(x, y, p)$ and the negation would persist to W^* . Hence the preservation case applies. Let y_0 be the chosen preservation witness. Lemma 4.11 gives

$$W^* \models \varphi_m(x, y_0, p).$$

Functionality of $A_{m,p}$ and the assumption $W^* \models \varphi_m(x, y, p)$ imply $y_0 = y$. Therefore the construction introduces, or has already introduced, the code $\text{Code}(p, x, y, m)$. By Lemma 5.1, the relevant tail has allowable countable initial segments, so the persistence and non-interference clauses of Lemma 3.4 give $W^* \models \Phi(p, x, y, m)$.

The two implications prove the equivalence. \square

Theorem 5.8. *In W^* , every total Π_3^1 -function has a Σ_3^1 graph.*

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a total function in W^* whose graph is Π_3^1 . Choose $m < \omega$ and a real parameter p such that

$$\text{Graph}(f) = A_{m,p} = \{(x, y) : \varphi_m(x, y, p)\}.$$

By Lemma 5.7, for all reals x, y in W^* ,

$$(x, y) \in \text{Graph}(f) \iff \Phi(p, x, y, m).$$

Since Φ is uniformly Σ_3^1 in the parameter p , the graph of f is $\Sigma_3^1(p)$. \square

6 Combining with Π_3^1 -reduction

We finally explain how the preceding construction is combined with the forcing construction for Π_3^1 -reduction from [6]. We use only the fixed point construction for reduction and its closure properties. The erroneous assertion is Lemma 3.13 in the last part of [6], concerning graphs of partial Π_3^1 functions. This lemma in fact does not hold in $W[G_{\omega_1}]$, contrary to what the lemma claims¹. It is replaced by the total-graph theorem proved in Sections 4 and 5.

¹Indeed, the proof erroneously claims that under the lemmas assumptions case 1 (i) applies in the iteration. A short look at the definition reveals that in fact case 1 (ii) must apply which implies that lemma 3.13 is not true as stated. The intuition to consider Π_3^1 graphs of functions and their Σ_3^1 -definability was the right one though, as this section shows. The lemma 3.13 can never hold as it would collapse the projective hierarchy.

Fix, in addition to the enumeration $\langle \varphi_m(v_0, v_1, v_2) \mid m < \omega \rangle$ of binary Π_3^1 formulas used above, a recursive enumeration

$$\langle \theta_m(u, v) \mid m < \omega \rangle$$

of all lightface Π_3^1 formulas in two real variables, chosen so that

$$\theta_0(u, v) \equiv u = v.$$

For a real parameter p , write

$$B_{m,p} = \{x \in \mathbb{R} \mid \theta_m(x, p)\}.$$

Thus every boldface Π_3^1 set of reals is some $B_{m,p}$.

We refine the fixed definable independent sequence of coding trees into three definable pairwise disjoint independent subsequences

$$\vec{S}^{\text{gr}}, \quad \vec{S}^{\text{red},1}, \quad \vec{S}^{\text{red},2}.$$

This is just a definable thinning of \vec{S} . The graph construction of the previous sections is now read as using only \vec{S}^{gr} ; its decoding predicate will be denoted by

$$\Phi^{\text{gr}}(p, x, y, m).$$

The reduction construction uses the two sequences $\vec{S}^{\text{red},1}$ and $\vec{S}^{\text{red},2}$. We write

$$\Psi_i(p, x, m, k) \quad (i = 1, 2)$$

for the usual Σ_3^1 statement that the real coding (p, x, m, k) has been written into the i -th reduction sequence. All coordinate families, reservoir coordinates and almost-disjoint coding blocks used for the three sequences are chosen disjointly.

The part of [6] needed here can be isolated as follows.

Proposition 6.1 (The reduction fixed point). *The reduction construction of [6, Section 3], applied to $\vec{S}^{\text{red},1}$ and $\vec{S}^{\text{red},2}$ and with real parameters coded into the reals handled by the bookkeeping, has the following properties.*

- (i) *It gives a derivative hierarchy of hybrid coding forcings which preserves ω_1 , is closed under regular subforcings, quotients by regular subforcings, cones, countable concatenations and products with disjoint coordinates, and stabilizes at a nonempty fixed point.*
- (ii) *The proof of reduction is insensitive to adding further disjoint coding coordinates, provided that the preservation tests in the reduction rule quantify over the full class of allowable future continuations under consideration.*
- (iii) *In an exhaustive ω_1 -length construction at the fixed point, for all p and all m, k with $m, k \neq 0$, the sets*

$$D_{m,k,p}^1 = B_{m,p} \cap \{x \mid \neg \Psi_1(p, x, m, k)\}$$

and

$$D_{m,k,p}^2 = B_{k,p} \cap \{x \mid \neg \Psi_2(p, x, m, k)\}$$

are $\Pi_3^1(p)$ and reduce the pair $(B_{m,p}, B_{k,p})$.

Proof. This is the fixed point proof of the Π_3^1 -reduction theorem from [6]. The additional parameter p is simply included in the real coded by the bookkeeping. Clause (ii) is exactly the role played by allowability in that proof: once the definition of the protected side of a reduction requirement quantifies over all allowed tails, later allowed forcing cannot create the bad configuration which the coding decision was designed to avoid. Disjoint additional coding coordinates do not change the decoding statements Ψ_i . \square

We now form one fixed point construction containing both kinds of requirements. The important point is that the word “allowable” is interpreted globally: the total-graph preservation derivative quantifies over future reduction actions as well as future graph actions, and the reduction construction similarly tests preservation against future graph actions.

Definition 6.2 (Combined derivative). *Let M be one of the local models obtained from a regular support. The classes*

$$\Gamma_\alpha^{M,\text{cmb}}$$

of combined α -allowable forcings over M are defined by the same local recursion as in Section 4, with the following two kinds of nontrivial successor stages.

At a stage, the bookkeeping first presents a regular support and names supported by it. After evaluating these names in the corresponding local model $N = M[G_A]$, one of the following rules is applied.

- (a) **Reduction requirement.** *The bookkeeping presents data (p, x, m, k) with $m, k \neq 0$. The stage follows the successor rule of the reduction construction from Proposition 6.1, using the reduction coding predicates Ψ_1, Ψ_2 and the reduction coordinates $\vec{S}^{\text{red},1}, \vec{S}^{\text{red},2}$. Whenever that rule refers to δ -allowable future forcings, it means forcings in $\Gamma_\delta^{N,\text{cmb}}$.*
- (b) **Total-graph requirement.** *The bookkeeping presents data (p, x, z, m) . Define, in N ,*

$$Y_{N,x,p,m}^{\delta,\text{cmb}} = \left\{ y \in N \cap \mathbb{R} \mid N \models \varphi_m(x, y, p) \text{ and } \forall \mathbb{R} \in \Gamma_\delta^{N,\text{cmb}} \mathbb{R} \Vdash_N \varphi_m(\check{x}, \check{y}, \check{p}) \right\}.$$

If this set is nonempty, choose its representative by the fixed well-order and write (p, x, y, m) into the graph coordinates, i.e. force with $\text{Code}^{\text{gr}}(p, x, y, m)$, and record the corresponding preservation entry. If the set is empty, write the bookkeeping guess (p, x, z, m) into the graph coordinates and record a guessing entry. This is precisely the successor rule from Definition 4.3, with Γ_δ replaced by $\Gamma_\delta^{\text{cmb}}$ and with the reserved graph coding coordinates.

At limit ordinals we take intersections of the preceding classes, locally over each M .

Lemma 6.3. *The combined hierarchy is definable, monotone, nonempty, closed under regular local tails, cones, countable concatenations and products with disjoint coordinates, and locally stabilizes. We denote the stable class over M by*

$$\Gamma_\infty^{M, \text{cmb}}.$$

Moreover, an ω_1 -length combined construction whose countable initial segments belong to the stable class preserves ω_1 and satisfies the same CH conclusion as in Lemma 3.4.

Proof. The proof is the simultaneous induction from Section 4, with one extra disjoint family of requirements. The graph stages are handled by the closure lemmas of Section 4; the reduction stages are handled by Proposition 6.1. The coding coordinates for the three families are disjoint, so products and subforcings are obtained by taking the corresponding products and restrictions in each family separately and then interleaving the resulting presentations. Monotonicity and nonemptiness are immediate from the two component constructions. Stabilization follows because, up to the fixed coding conventions, all forcings involved have size at most \aleph_1 , so the decreasing hierarchy of sets of possible presentations must eventually be constant. The preservation of ω_1 and the CH conclusion are the same hybrid forcing argument as in Lemma 3.4. \square

Let $\Gamma_\infty^{\text{cmb}}$ be the stable class over the initial model W . We now run an ω_1 -length construction over W with exhaustive bookkeeping for both kinds of requirements.

At reduction stages, with data (p, x, m, k) and $m, k \neq 0$, we perform the final reduction action from [6], interpreted with $\Gamma_\infty^{\text{cmb}}$ as the class of allowable continuations and using only the reduction coordinates. At total-graph stages, with data (p, x, z, m) , we perform the final action from Section 5, again with $\Gamma_\infty^{\text{cmb}}$ in place of Γ_∞ and using only the graph coordinates. Thus a no-witness graph stage first introduces the guessing code and then destroys the guessed value by a combined ∞ -allowable tail when necessary; if a later preservation witness appears, the squaring continuation is taken inside the combined stable class.

Let $G_{\omega_1}^{\text{cmb}}$ be generic for this final construction and put

$$W^{\text{cmb}} = W[G_{\omega_1}^{\text{cmb}}].$$

Lemma 6.4. *Every countable initial segment of the final combined construction is $\Gamma_\infty^{\text{cmb}}$ -allowable.*

Proof. At a reduction stage this is part of the reduction fixed point construction from Proposition 6.1, now read with combined allowable tails. At a graph stage it is the argument of Lemma 5.1, with every occurrence of Γ_∞ replaced by $\Gamma_\infty^{\text{cmb}}$. The appended destruction tails and the squaring tails are combined ∞ -allowable by Lemma 6.3. Countable concatenations are handled by the same lemma. \square

Lemma 6.5. *In W^{cmb} , the Π_3^1 -reduction property holds.*

Proof. Let $B_{m,p}$ and $B_{k,p}$ be two boldface $\mathbf{\Pi}_3^1$ sets of reals. If one of the indices is 0, then one of the two sets is all of \mathbb{R} , and the reduction is trivial. So assume $m, k \neq 0$.

The construction meets the reduction requirement for (p, m, k) cofinally often. At such stages the old reduction rule is applied, but its preservation tests quantify over the combined future forcings. Hence any protection obtained in the reduction construction is protection against later graph stages as well. Since the graph coordinates are disjoint from the reduction coordinates, they neither create nor destroy any decoding statement $\Psi_i(p, x, m, k)$. Thus the proof of Proposition 6.1 applies verbatim in the final combined extension.

Consequently

$$D_{m,k,p}^1 = B_{m,p} \cap \{x \mid \neg\Psi_1(p, x, m, k)\}$$

and

$$D_{m,k,p}^2 = B_{k,p} \cap \{x \mid \neg\Psi_2(p, x, m, k)\}$$

are $\mathbf{\Pi}_3^1(p)$ and reduce $(B_{m,p}, B_{k,p})$. \square

Lemma 6.6. *In W^{cmb} , if $A_{m,p}$ is the graph of a total function, then for all reals x, y ,*

$$A_{m,p}(x, y) \iff \Phi^{\text{gr}}(p, x, y, m).$$

In particular, every total $\mathbf{\Pi}_3^1$ graph is Σ_3^1 .

Proof. The proof of Lemma 5.7 is local to the graph coordinates once the class of allowable tails has been fixed. In the present construction that class is the combined class $\Gamma_\infty^{\text{cmb}}$. Thus a graph preservation witness is preserved not merely against future graph coding, but also against future reduction coding. The fixed-point dichotomy, the clearing of old candidates, the ‘‘old reals do not become protected’’ lemma, and the no-witness squaring lemma all use only closure under cones, products, regular tails and concatenations; these are supplied by Lemma 6.3. The product-name separation principle is unaffected by adding disjoint reduction coordinates.

For completeness we recall the decisive point. A graph guessing code for (p, x, w, m) only records that the combined preservation set for (m, p, x) was empty in the earlier local model. If the final graph is total and its value at x is y , then either $w = y$, in which case the earlier destruction of the guessed value gives a persistent contradiction, or $w \neq y$, in which case a later preservation action for y yields, by the same squaring argument, two distinct combined-preserved witnesses in the x -section. Hence no guessing code can contribute to a total graph. The first-stage and persistence properties of the graph decoding then give exactly the equivalence displayed above. \square

Theorem 6.7. *Assuming $\text{Con}(\text{ZFC})$, it is consistent that $\mathbf{\Pi}_3^1$ -reduction holds, $\mathbf{\Pi}_3^1$ -uniformization fails, and every total $\mathbf{\Pi}_3^1$ -function has a Σ_3^1 graph.*

Proof. The final combined construction preserves ω_1 and yields a model of ZFC by Lemmas 3.4, 6.3 and 6.4. Lemma 6.5 gives Π_3^1 -reduction. Lemma 6.6 gives the total-graph property. Finally, Corollary 2.2 implies that Π_3^1 -uniformization fails in this model. \square

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