Projective maximal families of orthogonal measures and global Σ-uniformization

Stefan Hoffelner *

15.6.2026

Abstract

We study the interaction between projective uniformization and definable maximal orthogonal families of Borel probability measures on 2^ω. Starting from L, we build two forcing extensions in both of which there is a Π^1_2 maximal orthogonal family and there are no boldface $\mathbf{\Sigma}^1_2$ maximal orthogonal families, and every Σ^1_n relation admits a Σ^1_n uniformization for all $n \geq 2$. Additionally in the first model $\mathfrak{b} = \mathfrak{c} = \aleph_3$ holds; in the second $\mathfrak{b} = \aleph_1$ and $\mathfrak{c} = \aleph_2$ holds.

Keywords. Projective uniformization; maximal orthogonal families; projective wellorders; definable coding; cardinal characteristics.

MSC 2020. 03E15, 03E35, 03E40, 03E45.

1 Introduction

Let $\mathcal{P}(2^\omega)$ be the Polish space of Borel probability measures on 2^ω. Two measures $\mu, \nu \in \mathcal{P}(2^\omega)$ are orthogonal, written $\mu \perp \nu$, if there is a Borel set $B \subseteq 2^\omega$ such that

$$\mu(B) = 0 \qquad \text{and} \qquad \nu(2^\omega \setminus B) = 0.$$

A family $\mathcal{A} \subseteq \mathcal{P}(2^\omega)$ is a maximal orthogonal family, or an m.o. family, if its elements are pairwise orthogonal and it is maximal with respect to this property.

The projective complexity of m.o. families is a sensitive test of the structure of the real line. Preiss and Rataj proved that there are no analytic maximal orthogonal families [PR85]. Fischer and Törnquist showed that this lower bound is sharp in L, by constructing a coanalytic maximal orthogonal family [FT10]. At the same time, their result revealed a genuine obstruction: if there is a Cohen real over L, then there is no Π^1_1 maximal orthogonal family; in its relativized form, the presence of Cohen reals over all models $L[a]$ rules out boldface $\mathbf{\Pi}^1_1$ maximal orthogonal families. Thus the existence of a low-level definable m.o. family is not merely a local definability phenomenon; it reflects global information about the ambient real line.

Fischer–Friedman–Törnquist showed that the obstruction at the coanalytic level does not persist unchanged at the next projective level. They constructed forcing extensions in which there is a Π^1_2 maximal orthogonal family, there is no boldface $\mathbf{\Sigma}^1_2$ maximal orthogonal family, and either

$$\mathfrak{b} = \mathfrak{c} = \omega_3 \qquad \text{or} \qquad \mathfrak{b} = \omega_1,\ \mathfrak{c} = \omega_2;$$

*The author's research was funded in whole by the Austrian Science Fund (FWF) Grant-DOI 10.55776/P37228.

see [FFT12]. These models also carry the Fischer–Friedman–Zdomskyy projective coding apparatus, in particular a Δ^1_3 wellorder of the reals. The result belongs to the broader line of work on projective wellorders, definable maximal families and cardinal characteristics developed in [JS70, Har77, FF10, FFZ11, FFK14b, FFK13, FFK14a].

There is also a natural link between maximal orthogonal families and uniformization. The obstruction arguments do not use regularity alone. They first turn maximality into a definable choice problem, and then apply regularity to the resulting invariant assignment. In Proposition 2.2 below we isolate this point in the Σ^1_2 case: the standard uniformization step yields a Δ^1_2 countable-valued invariant for the product-measure equivalence relation, and turbulence rules this out under the relevant regularity assumption. Thus uniformization already occurs in the negative part of the theory, while the forcing constructions below show that global Σ-uniformization is compatible with a Π^1_2 maximal orthogonal family.

The purpose of the present paper is to show that the compatibility phenomenon is stronger than the earlier results indicate. A Π^1_2 maximal orthogonal family can coexist not only with prescribed values of cardinal characteristics and the Fischer–Friedman–Zdomskyy coding machinery, but also with global projective uniformization at every Σ-level. For a projective pointclass Γ, let Γ-UP denote the assertion that every set $A \subseteq 2^\omega \times 2^\omega$ in Γ has a uniformization whose graph is again in Γ; that is, there is a set $U \subseteq A$ in Γ which is the graph of a function and whose domain is $\operatorname{proj}(A)$. We write

$$\mathsf{UP}_\Sigma \quad \text{for} \quad \forall n \geq 2\ (\Sigma^1_n\text{-}\mathsf{UP}).$$

The case $n = 2$ follows from Kondo's theorem; see, for example, [Kec95, Mos09]. The forcing constructions below therefore arrange the cases $n \geq 3$.

Classical routes to projective uniformization proceed either from good projective wellorders, beginning with Addison's theorem [Add59], or from determinacy and large cardinals, as in work of Moschovakis, Martin–Steel and Woodin [Mos71, MS89]. The method here is different. We use the same Fischer–Friedman–Zdomskyy stationary-set coding machinery which underlies the Π^1_2 definition of the maximal orthogonal family as the base coding predicate for global Σ-uniformization. At a uniformization stage, the construction codes a compressed record of the matrix information needed to select the least successful witness and to exclude later candidates. This use of projective predicates to code truth values of finite patterns of lower-level statements follows the method developed in [Hof25b] and refined in [Hof25c], [Hof26] and [Hof25a]. In this way the final models show that a low-level projective m.o. family is compatible with a strong global projective choice principle.

The main results are the following two consistency theorems. Throughout, "no boldface $\boldsymbol{\Sigma}^1_2$ maximal orthogonal family" means that there is no $\Sigma^1_2(a)$ maximal orthogonal family for any real parameter a.

Theorem 1.1. *Assume $V = L$. There is a forcing extension in which*

$$\mathfrak{b} = \mathfrak{c} = \omega_3,$$

there is a Π^1_2 maximal orthogonal family of Borel probability measures on 2^ω, there are no boldface $\boldsymbol{\Sigma}^1_2$ maximal orthogonal families, and UP_Σ holds.

Theorem 1.2. *Assume $V = L$. There is a forcing extension in which*

$$\mathfrak{b} = \omega_1, \qquad \mathfrak{c} = \omega_2,$$

there is a Π^1_2 maximal orthogonal family of Borel probability measures on 2^ω, there are no boldface $\boldsymbol{\Sigma}^1_2$ maximal orthogonal families, and UP_Σ holds.

The two models use the same abstract organization but different preservation technology. In both constructions the coding coordinates are split into disjoint reservoirs. The MO-reservoir is used to run the Fischer–Friedman–Törnquist maximal-orthogonal-family construction. The Σ-reservoir is used for the global uniformization construction. The separation of the reservoirs, together with the no-unwanted-codes theorem, prevents the two kinds of projective codes from interfering.

For Theorem 1.1 we use the finite-support σ-centered Fischer–Friedman–Zdomskyy construction over the large-continuum coding blocks, together with the standard bookkeeping which forces $\mathfrak{b} = \mathfrak{c} = \omega_3$. For Theorem 1.2 we replace this by the countable-support S-proper, ω^ω-bounding implementation of the same coding package, interleaved with random forcing, obtaining $\mathfrak{b} = \omega_1$ and $\mathfrak{c} = \omega_2$.

The paper is organized as follows. The next section records the basic facts about orthogonality and the obstruction to Σ^1_2 maximal orthogonal families used in the final models. Section 3 recalls the Fischer–Törnquist coding of reals into non-atomic measures. Section 4 records the Fischer–Friedman–Zdomskyy stationary-set coding machinery and the no-unwanted-codes principle used later. The large-continuum construction and the proof of Theorem 1.1 are given in Section 5. The countable-support construction proving Theorem 1.2 is given in Section 7.

2 Preliminaries on maximal orthogonal families

We collect the facts about orthogonal measures which are used in the forcing argument. The aim is not to reprove the Fischer–Friedman–Törnquist construction, but to record the definability facts and the obstruction theorem which will be used later.

Let $X = 2^\omega$ and let $\mathcal{P}(X)$ denote the Polish space of Borel probability measures on X, equipped with the usual weak topology. We write $\mu \perp \nu$ if μ and ν are mutually singular, and we write $\mu \not\perp \nu$ otherwise. Thus $\mu \perp \nu$ means that there is a Borel set $B \subseteq X$ such that

$$\mu(B) = 0 \qquad \text{and} \qquad \nu(X \setminus B) = 0.$$

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is orthogonal if any two distinct members of $\mathcal{A}$ are mutually singular. It is a maximal orthogonal family if it is orthogonal and every $\mu \in \mathcal{P}(X)$ is non-orthogonal to at least one member of $\mathcal{A}$. Since point measures form a closed pairwise orthogonal family, and every point measure is orthogonal to every non-atomic measure, the forcing constructions build the non-atomic part of the family and then add the point measures at the end. Atomic measures are then covered by the point masses at their atoms.

We shall use the following standard coding fact for the usual recursive presentation of the space of Borel probability measures on 2^ω.

Lemma 2.1 (Fischer–Törnquist)**.** *In the presentation*

$$p(2^\omega) = \{f : 2^{<\omega} \to [0,1] : f(\emptyset) = 1 \textit{ and } f(s) = f(s^\frown 0) + f(s^\frown 1) \textit{ for all } s \in 2^{<\omega}\},$$

with $f \mapsto \mu_f$, the following relations are arithmetical in the measure codes:

$$f \in p_c(2^\omega), \qquad \mu_f \perp \mu_g, \qquad \mu_f \not\perp \mu_g, \qquad \mu_f \ll \mu_g, \qquad \mu_f \approx \mu_g.$$

Consequently these relations may be inserted into projective definitions without changing the projective level.

This is the coding lemma from [FT10]. We use it only as a definability fact.

We next record the uniformization step behind the standard obstruction to low-complexity maximal orthogonal families. The proof is the Fischer–Törnquist and Fischer–Friedman–Törnquist turbulence argument, written so as to display where uniformization enters.

Proposition 2.2. *Let $a \in 2^\omega$. Suppose that every $\Delta^1_2(a)$ subset of a Polish space has the Baire property, respectively is Lebesgue measurable. Then there is no $\Sigma^1_2(a)$ maximal orthogonal family of Borel probability measures on 2^ω.*

Proof sketch. We sketch the Baire-property case, following the obstruction argument of Fischer–Törnquist and Fischer–Friedman–Törnquist. The measure case is proved in [FFT12, Proposition 1]; it uses the same idea together with the standard reduction of E_I to its restriction to a positive-measure closed set.

Let E_I be the equivalence relation on 2^ω defined by

$$x E_I y \iff \sum_{n<\omega} \frac{|x(n) - y(n)|}{n+1} < \infty.$$

The product-measure construction of Kechris–Sofronidis, recalled by Fischer–Törnquist, gives a recursive continuous map

$$x \longmapsto \mu^x$$

from 2^ω into the non-atomic probability measures on 2^ω. By Kakutani's dichotomy theorem for infinite product measures [Kak48], this map satisfies

$$x E_I y \Rightarrow \mu^x \approx \mu^y, \qquad \neg(x E_I y) \Rightarrow \mu^x \perp \mu^y.$$

The equivalence relation E_I is the turbulent relation used in the Preiss–Rataj–Kechris–Sofronidis obstruction; see [KS01, FT10].

Suppose toward a contradiction that $\mathcal{A}$ is a $\Sigma^1_2(a)$ maximal orthogonal family. For each $x \in 2^\omega$, let

$$\mathcal{A}_x = \{\nu \in \mathcal{A} : \nu \not\perp \mu^x\}.$$

Maximality gives $\mathcal{A}_x \neq \emptyset$. The c.c.c.-below property of absolute continuity gives that $\mathcal{A}_x$ is countable; this is the same countability fact used in [KS01, FT10]. Define

$$Q \subseteq 2^\omega \times \mathcal{P}(2^\omega)^\omega$$

by declaring that $Q(x, \vec{\nu})$ holds iff $\vec{\nu}$ enumerates precisely the members of $\mathcal{A}$ which are not orthogonal to μ^x. Since $\mathcal{A}$ is $\Sigma^1_2(a)$ and the relevant relations between measure codes are arithmetical by Lemma 2.1, the relation Q is $\Sigma^1_2(a)$, and every section Q_x is nonempty.

By Kondo's uniformization theorem, applied to this $\Sigma^1_2(a)$ relation, there is a total function

$$f : 2^\omega \longrightarrow \mathcal{P}(2^\omega)^\omega$$

whose graph is $\Sigma^1_2(a)$ and such that $Q(x, f(x))$ for all x. Since the graph is total and functional, it is in fact $\Delta^1_2(a)$. Hence, under the Baire-property hypothesis for $\Delta^1_2(a)$ sets, the function f is Baire measurable.

Put

$$F(x) = \{f(x)(n) : n < \omega\}.$$

If $x E_I y$, then $\mu^x \approx \mu^y$, and therefore

$$\nu \not\perp \mu^x \iff \nu \not\perp \mu^y$$

for every measure ν. Hence $F(x) = F(y)$ whenever $x E_I y$.

By the generic ergodicity theorem for this turbulent relation [KS01], the Baire measurable invariant assignment $x \mapsto F(x)$ is constant on a comeager set. Thus there are a comeager set $C \subseteq 2^\omega$ and a fixed countable family

$$B = \{\nu_n : n < \omega\} \subseteq \mathcal{P}(2^\omega)$$

such that

$$F(x) = B \qquad \text{for all } x \in C.$$

Since $F(x)$ enumerates the members of $\mathcal{A}$ not orthogonal to μ^x, for every $x \in C$ there is some $n < \omega$ such that

$$\nu_n \not\perp \mu^x.$$

Hence

$$C \subseteq \bigcup_{n<\omega} \{x \in 2^\omega : \nu_n \not\perp \mu^x\}.$$

For each fixed measure ν, the set

$$\{x \in 2^\omega : \nu \not\perp \mu^x\}$$

is meager by the product-measure lemma of Fischer–Törnquist [FT10, Lemma 2.4]. The right-hand side is therefore meager, contradicting that C is comeager. This proves the Baire-property case. The Lebesgue-measurable/random version is the corresponding measure-case argument from [FFT12, Proposition 1]. □

The form used in the forcing arguments is the following consequence.

Corollary 2.3. *Let $a \in 2^\omega$. If there is either a Cohen real over $L[a]$ or a random real over $L[a]$, then there is no $\Sigma^1_2(a)$ maximal orthogonal family of Borel probability measures on 2^ω.*

Proof. This is [FFT12, Proposition 1]. We recall how it follows from the preceding obstruction principle. The regularity theorems of Judah–Shelah [JS89] imply that a Cohen real over $L[a]$ gives the Baire property for $\Delta^1_2(a)$ sets, and that a random real over $L[a]$ gives the corresponding Lebesgue measurability statement. Proposition 2.2 then rules out a $\Sigma^1_2(a)$ maximal orthogonal family. □

This corollary is the only obstruction theorem about Σ^1_2 maximal orthogonal families used below. In the first model we arrange, for every real parameter a, a Cohen real over $L[a]$; in the second model we arrange a random real over $L[a]$.

3 Fischer–Törnquist coding of reals into measures

We now recall the part of the Fischer–Törnquist coding which is used at the maximal-orthogonal-family stages. This section is only a coding summary. The proof of the coding lemma is not repeated here.

Let

$$p(2^\omega) = \{f : 2^{<\omega} \to [0,1] : f(\emptyset) = 1 \text{ and } f(s) = f(s^\frown 0) + f(s^\frown 1) \text{ for all } s \in 2^{<\omega}\}.$$

The map $f \mapsto \mu_f$, where $\mu_f(N_s) = f(s)$ for basic clopen sets N_s, is a recursive presentation of the Polish space of Borel probability measures on 2^ω. We write $p_c(2^\omega)$ for the arithmetical set of codes for non-atomic measures. By Lemma 2.1, the relations $\perp, \not\perp$, absolute continuity and absolute equivalence may be treated arithmetically in these codes.

For $f \in p_c(2^\omega)$ and $s \in 2^{<\omega}$, let $t(s, f)$ be the lexicographically least $t \supseteq s$ such that

$$f(t^\frown 0) > 0 \qquad \text{and} \qquad f(t^\frown 1) > 0,$$

if such a t exists, and put $t(s, f) = \emptyset$ otherwise. Define recursively

$$t^f_0 = \emptyset, \qquad t^f_{n+1} = t((t^f_n)^\frown 0, f).$$

Since f is non-atomic, the sequence $(t_n^f : n < \omega)$ is recursive in f and strictly increases in length. It determines a canonical branch through the support of μ_f along which one may write binary information without changing the measure class.

Define a relation $R \subseteq p_c(2^\omega) \times 2^\omega$ by declaring that $R(f, z)$ holds if and only if for every $n < \omega$,

$$z(n) = 1 \iff f((t_n^f)^\frown 0) = \frac{2}{3} f(t_n^f) \text{ and } f((t_n^f)^\frown 1) = \frac{1}{3} f(t_n^f),$$
$$z(n) = 0 \iff f((t_n^f)^\frown 0) = \frac{1}{3} f(t_n^f) \text{ and } f((t_n^f)^\frown 1) = \frac{2}{3} f(t_n^f).$$

When $R(f, z)$ holds, we say that the measure code f *Fischer–Törnquist-codes* the real z. The domain

$$\mathrm{dom}(R) = \{f \in p_c(2^\omega) : (\exists z) R(f, z)\}$$

is arithmetical, in fact Π^0_1, and the decoding map $r : \mathrm{dom}(R) \to 2^\omega$ given by $r(f) = z$ iff $R(f, z)$ is arithmetical on its domain.

Lemma 3.1. *There is a recursive function*

$$G : p_c(2^\omega) \times 2^\omega \longrightarrow p_c(2^\omega)$$

such that for every $f \in p_c(2^\omega)$ and every $z \in 2^\omega$,

$$\mu_{G(f,z)} \approx \mu_f \qquad \textit{and} \qquad R(G(f, z), z).$$

Thus $G(f, z)$ is a canonical code for a measure in the same measure class as μ_f, and this new code remembers z by the relation R.

Proof. This is the coding lemma of Fischer–Törnquist, used in [FT10] and in the Fischer–Friedman–Törnquist construction [FFT12]. We use it as a black box. □

The forcing application is as follows. At a maximal-orthogonal-family stage one has a real x which codes a non-atomic measure μ_x and which, in the current intermediate model, is orthogonal to the family constructed so far. The stage first writes x into the Fischer–Friedman–Zdomskyy almost-disjoint/stationary-set coding apparatus. Let u be the real produced by this local coding. The real u is an ordinary real, but it carries the local stationary-set pattern from which suitable countable transitive models recover the original measure code x. The actual measure inserted into the family is then

$$\mu_{G(x,u)}.$$

This measure is equivalent to μ_x, and its code $G(x, u)$ Fischer–Törnquist-codes u. Therefore membership in the final family can be read in two steps:

$$G(x, u) \xrightarrow{\text{FT-decode}} u \xrightarrow{\text{local a.d./stationary decode}} x.$$

The equality $r = G(x, u)$ is part of the intended membership test. The weaker statement $\mu_r \approx \mu_x$ would not be sufficient, since it would allow several equivalent measure codes into the same orthogonal family. The recursiveness of G is what makes the equality test absolute to the countable suitable models used in the final Π^1_2 definition.

4 The Fischer–Friedman–Zdomskyy coding machinery

We now summarize the forcing-theoretic coding package used by Fischer–Friedman–Törnquist. This is the same large-continuum coding package introduced by Fischer–Friedman–Zdomskyy for projective wellorders and mad families, rather than the later Suslin-tree version used for Martin's Axiom. In this section we spell out the preliminary forcing, using the notation of Fischer–Friedman–Zdomskyy. The point of the package is to provide a projectively recognizable way of saying that a real was intentionally written into a prescribed block of the stationary-set coding apparatus, together with a preservation theorem saying that no later forcing can create such a code accidentally.

4.1 The stationary reservoirs

Work over L. Fix a $\diamondsuit_{\omega_2}(\mathrm{cof}(\omega_1))$-sequence

$$\langle G_\xi : \xi \in \omega_2 \cap \mathrm{cof}(\omega_1)\rangle$$

which is Σ_1-definable over L_{ω_2}. For every $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 coding the ordinal α. Define

$$S^\alpha = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset\}.$$

Then

$$\vec{S} = \langle S^\alpha : \alpha < \omega_3\rangle$$

is a uniformly definable sequence of stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega_1)$, and the members of the sequence are mutually almost disjoint: if $\alpha \neq \beta$, then $S^\alpha \cap S^\beta$ is bounded in ω_2. Also put

$$S^{-1} = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = \emptyset\},$$

a stationary set disjoint from every S^α.

A transitive model M of ZF^- is called *suitable* if ω_3^M exists and

$$\omega_3^M = \omega_3^{L^M}.$$

Then also $\omega_1^M = \omega_1^{L^M}$ and $\omega_2^M = \omega_2^{L^M}$. The final projective definitions quantify over real codes for countable suitable transitive models. Once such a model is fixed, all the decoding assertions below are first-order assertions over the coded model.

4.2 Step 0: killing the stationary sets

For each $\alpha < \omega_3$, let P^0_α be the forcing whose conditions are bounded closed sets

$$c \subseteq \omega_2 \qquad \text{with} \qquad c \cap S^\alpha = \emptyset,$$

ordered by end-extension. If G^0_α is P^0_α-generic, then

$$C_\alpha = \bigcup G^0_\alpha$$

is a club subset of ω_2 disjoint from S^α. Let

$$P^0 = \prod_{\alpha<\omega_3} P^0_\alpha$$

be the direct product with supports of size at most ω_1. The forcing P^0 is countably closed, ω_2-distributive and has the ω_3-chain condition. The ω_2-distributivity uses the auxiliary stationary set S^{-1}.

4.3 Step 1: coding the kills down to ω_1

In this paragraph, as in Fischer–Friedman–Zdomskyy, 0 is treated as a limit ordinal. For a set X of ordinals write

$$0(X) = \{\eta : 3\eta \in X\}, \qquad I(X) = \{\eta : 3\eta + 1 \in X\}, \qquad II(X) = \{\eta : 3\eta + 2 \in X\}.$$

After forcing with P^0, for each $\alpha < \omega_3$ choose a set $D_\alpha \subseteq \omega_2$ coding the tuple

$$\langle C_\alpha, W_\alpha, W_\gamma \rangle,$$

where γ is the largest limit ordinal with $\gamma \le \alpha$. More explicitly, D_α is chosen so that

$$0(D_\alpha) = C_\alpha, \qquad I(D_\alpha) = W_\alpha, \qquad II(D_\alpha) = W_\gamma.$$

Let E_α be the club in ω_2 consisting of all intersections $N \cap \omega_2$, where

$$N \prec L_{\alpha+\omega_2+1}[D_\alpha] \qquad \text{and} \qquad \omega_1 \cup \{D_\alpha\} \subseteq N.$$

For a set X of ordinals let

$$\mathrm{Even}(X) = \{\eta : 2\eta \in X\}.$$

Choose $Z_\alpha \subseteq \omega_2$ such that

$$\mathrm{Even}(Z_\alpha) = D_\alpha,$$

and such that whenever $\beta < \omega_2$ is ω_2^M for some suitable model M and $Z_\alpha \cap \beta \in M$, then $\beta \in E_\alpha$. This is the Fischer–Friedman–Zdomskyy condensation device: between adjacent points of E_α one places enough information to make any suitable model containing an initial segment of Z_α recognize that its height is an E_α-point.

The consequence is the following absoluteness property, denoted $(*)_\alpha$ in [FFZ11]. If $\beta < \omega_2$ and M is suitable with

$$\omega_1 \subseteq M, \qquad \omega_2^M = \beta, \qquad Z_\alpha \cap \beta \in M,$$

then

$$M \models \psi(\omega_2, Z_\alpha \cap \beta),$$

where $\psi(\omega_2, X)$ is the assertion:

> $\mathrm{Even}(X)$ codes a tuple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S^{\bar{\alpha}}$.

Here all symbols are interpreted inside M.

Next fix, using the standard diamond sequence on ω_1, a nicely definable almost disjoint sequence

$$\vec{A} = \langle A_\xi : \xi < \omega_2 \rangle$$

of stationary subsets of ω_1. In L^{P^0}, let P^1_α be the forcing of all pairs

$$\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1},$$

where

$$\langle t_0, t_1 \rangle \le \langle s_0, s_1 \rangle$$

iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$, and

$$(t_0 \setminus s_0) \cap A_\xi = \emptyset \qquad \text{for every } \xi \in s_1.$$

The generic object is denoted $X_\alpha \subseteq \omega_1$; it almost-disjointly codes Z_α using the sequence $\vec{A}$. Let

$$P^1 = \prod_{\alpha<\omega_3} P^1_\alpha$$

with countable supports. The forcing P^1 is countably closed and has the ω_2-chain condition.

The second absoluteness property is denoted $(**)_\alpha$ in [FFZ11]. If $\omega_1 < \beta \leq \omega_2$ and M is suitable with

$$\omega_2^M = \beta, \qquad \{X_\alpha\} \cup \omega_1 \subseteq M,$$

then

$$M \models \varphi(\omega_1, \omega_2, X_\alpha),$$

where $\varphi(\omega_1, \omega_2, X)$ says:

> Using the sequence $\vec{A}$, the set X almost-disjointly codes a subset $\bar{Z}$ of ω_2 whose even part $\mathrm{Even}(\bar{Z})$ codes a tuple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S^{\bar{\alpha}}$.

Again the assertion is internal to M.

4.4 Step 2: localization

We now force a localization of the X_α's. Suppose $X, X' \subseteq \omega_1$ are such that $\varphi(\omega_1, \omega_2, X)$ and $\varphi(\omega_1, \omega_2, X')$ hold in all suitable models with the correct ω_1 containing the relevant set. The Fischer–Friedman–Zdomskyy localization poset $L(X, X')$ consists of all functions

$$r : |r| \longrightarrow 2,$$

where $|r|$ is a countable limit ordinal, such that:

(1) if $\gamma < |r|$, then $\gamma \in X$ iff $r(3\gamma) = 1$;

(2) if $\gamma < |r|$, then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$;

(3) if $\gamma \leq |r|$, M is a countable suitable model containing $r \restriction \gamma$ as an element, and $\gamma = \omega_1^M$, then

$$M \models \varphi(\omega_1, \omega_2, X \cap \gamma) \wedge \varphi(\omega_1, \omega_2, X' \cap \gamma).$$

The order is end-extension. For every $\alpha \in \mathrm{Lim}(\omega_3)$ and $m \in \omega$, set

$$P^2_{\alpha+m} = L(X_{\alpha+m}, X_\alpha),$$

and let

$$P^2 = \prod_{\alpha \in \mathrm{Lim}(\omega_3)} \prod_{m \in \omega} P^2_{\alpha+m}$$

with countable supports. In $L^{P^0 * P^1}$ the forcing P^2 has the ω_2-chain condition. The generic for $P^2_{\alpha+m}$ is identified with a subset

$$Y_{\alpha+m} \subseteq \omega_1,$$

namely the characteristic function added by the localization forcing.

The localization property is denoted $(***)_\alpha$. For every $\beta < \omega_1$ and every suitable M such that

$$\omega_1^M = \beta \qquad \text{and} \qquad Y_{\alpha+m} \cap \beta \in M,$$

one has

$$M \models \varphi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \varphi(\omega_1, \omega_2, X_\alpha \cap \beta).$$

Thus $Y_{\alpha+m}$ lets countable suitable models recover both the local coordinate $X_{\alpha+m}$ and the base coordinate X_α.

Finally set

$$P_0 = P^0 * P^1 * P^2.$$

The main preservation fact for the preliminary forcing is:

Lemma 4.1. *The forcing $P_0 = P^0 * P^1 * P^2$ is ω-distributive. In particular, P_0 adds no reals.*

We use this lemma as a black box. The proof is the fusion argument of [FFZ11] over the least countable elementary submodel containing the initial condition, the forcing, and a prescribed sequence of dense sets. The point of the localization clause in $L(X, X')$ is precisely to make the lower bound in the P^2-coordinate a condition.

It is useful to record the deletion notation used in the no-unwanted-codes argument. For $\xi < \omega_3$, [FFZ11] writes $P^{0,,\xi}$, $P^{1,,\xi}$ and $P^{2,,\xi}$ for the products obtained by omitting the coordinate ξ from the corresponding preliminary forcing; for example

$$P^{0,,\xi} = \prod_{\alpha \in \omega_3 \setminus \{\xi\}} P^0_\alpha$$

with supports of size at most ω_1, and similarly for $P^{1,,\xi}$ and $P^{2,,\xi}$ with countable supports. Then

$$\widetilde{P}_0^{;\xi} := P^{0,,\xi} * P^{1,,\xi} * P^{2,,\xi} <_c P_0.$$

The quotient over this complete subforcing is the device which shows that, if the coordinate ξ is protected, then the final model has no real coding a club through ω_2 disjoint from S^ξ.

4.5 How the preliminary forcing is used later

After the preliminary forcing, the tail construction is a finite-support iteration of σ-centered forcings. Fix a nicely definable almost disjoint sequence

$$\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1,\ m < \omega \rangle$$

of subsets of ω. A coding stage chooses a fresh limit block α and a real to be coded z. The forcing at that stage has conditions (s_0, s_1), where s_0 is a finite initial segment of the new real and s_1 is a finite set of requirements drawn from

$$\bigcup_{m \in \Delta(z)} Y_{\alpha+m} \times \{m\}.$$

The order end-extends s_0 and requires the newly added part of s_0 to avoid $B_{\zeta,m}$ for every active requirement $(\zeta, m) \in s_1$. The generic real therefore almost-disjointly codes the sets $Y_{\alpha+m}$ for the indicated values of m. The map $z \mapsto \Delta(z) \subseteq \omega$ is chosen injectively, so the stationary-kill pattern recovers the coded real.

For a real r produced at such a stage, the intended decoding statement has the following form. For every countable suitable transitive M containing r, the model $(L[r])^M$ sees that, for some internal limit block $\bar{\alpha}$, the stationary sets

$$(S^{\bar{\alpha}+m})^M$$

are nonstationary for the integers m belonging to the prescribed pattern. The layered reason is:

$$r \quad \rightsquigarrow \quad Y_{\alpha+m} \cap \omega_1^M \quad \rightsquigarrow \quad X_{\alpha+m} \cap \omega_1^M,\ X_\alpha \cap \omega_1^M \quad \rightsquigarrow \quad Z_{\alpha+m}, Z_\alpha \quad \rightsquigarrow \quad C_{\alpha+m}.$$

Consequently the final coding predicates have the outer form

$$\forall M \big(M \text{ countable suitable and } r \in M \Rightarrow (L[r])^M \models \exists \bar{\alpha}\ \Theta(\bar{\alpha}, r)\big),$$

with Θ first-order over the coded model. This gives the desired Π^1_2 complexity.

The decisive preservation theorem is the no-unwanted-codes lemma. During the construction, whenever a block α is used to code a pattern, the complementary coordinates are placed into a protected set $A_\alpha \subseteq [\alpha, \alpha+\omega)$. If ξ belongs to the union of these protected coordinates, then in the final extension there is no real coding a stationary kill of S^ξ, i.e. there is no real from which one can construct a club in ω_2 disjoint from S^ξ. Equivalently, if the final model contains a real which appears, in all suitable countable transitive models, to code a stationary kill of S^ξ, then that kill must have been intentionally introduced at the corresponding coding stage. This is the lemma which turns the projective decoding predicate from a mere superset of the intended objects into an exact definition.

We will use this package with two disjoint coding tags, MO and Σ, and with a separate side-forcing reservoir. At maximal-orthogonal-family stages, the MO-tagged copy of the coding apparatus is used: the real produced by the Fischer–Friedman–Zdomskyy coding is then itself written into a measure code by the Fischer–Törnquist map G from Lemma 3.1. At global Σ-uniformization stages, the Σ-tagged copy of the same stationary-set package writes uniformization certificates. At cardinal-characteristic stages, the iteration adds the required reals without using either protected coding reservoir. Because the two reservoirs and the two real tags are disjoint, the no-unwanted-codes lemma applies separately to the MO-tagged and Σ-tagged coding predicates.

5 The large-continuum construction

This section records the construction of the first model. The construction is the Fischer–Friedman–Törnquist finite-support construction, but with the projective-wellorder stages replaced by global Σ-uniformization stages. The maximal-orthogonal-family stages and the cardinal-characteristic stages are kept. The proof of the first main theorem is packaged in Section 6.

5.1 The reservoirs

We work over L and first force with the preliminary Fischer–Friedman–Zdomskyy forcing

$$P_0 = P^0 * P^1 * P^2$$

from Section 4. Thus in the intermediate model after P_0 we have the stationary sequence

$$\vec{S} = \langle S^\xi : \xi < \omega_3 \rangle,$$

the localized sets $Y_{\lambda+m} \subseteq \omega_1$ for $\lambda \in \mathrm{Lim}(\omega_3)$ and $m < \omega$, and the no-unwanted-codes theorem for protected coordinates. A *coding block* is an interval

$$I_\lambda = \{\lambda + m : m < \omega\}$$

with $0 < \lambda \in \mathrm{Lim}(\omega_3)$. We choose the blocks so that distinct blocks are disjoint, equivalently if $\lambda < \lambda'$ are chosen block starts then $\lambda + \omega \leq \lambda'$. A block is *fresh at stage α* if no coordinate in I_λ has appeared in an earlier coding requirement, in an earlier protected set, or in the support of the $\mathbb{P}_\alpha$-names currently being acted on.

For later reference we make the support convention explicit. If $J \subseteq \omega_3$, let

$$P_0 \restriction J$$

denote the complete subforcing of the preliminary forcing in which all preliminary coordinates not indexed by J are trivial. Thus $P_0 \restriction \lambda$ uses only the coordinates $< \lambda$ of the three preliminary steps P^0, P^1, P^2. Once λ is above all block starts used before stage α, the already constructed finite-support iteration $\mathbb{P}_\alpha$ is read over $P_0 \restriction \lambda$: every earlier nontrivial coding iterand was defined from a block below λ, and the side iterands do not use fresh preliminary coordinates in or above I_λ.

A $P_0 * \dot{\mathbb{P}}_\alpha$-name $\dot{z}$ for a real is *supported below the block* λ if it is equivalent to a $P_0 \restriction \lambda * \dot{\mathbb{P}}_\alpha$-name. Equivalently, after replacing $\dot{z}$ by a nice name over the already constructed c.c.c. iteration, every condition occurring in the antichains deciding the bits of $\dot{z}$ has preliminary component trivial on I_λ and on every preliminary coordinate $\geq \lambda$. Its post-preliminary component may use coordinates $\beta < \alpha$ of the finite-support iteration, as any $\mathbb{P}_\alpha$-name may. Thus the phrase "iteration coordinates below α" refers only to the already constructed c.c.c. coordinates $\dot{\mathbb{Q}}_\beta$, $\beta < \alpha$; it does not allow the name to use any preliminary coordinate in the new block I_λ or above it. At a coding stage we choose λ above this preliminary support.

We partition the chosen block starts into three pairwise disjoint cofinal sets

$$\Lambda_\Sigma, \qquad \Lambda_{\mathsf{MO}}, \qquad \Lambda_{\mathfrak{b}}.$$

The first reservoir is used for global Σ-uniformization, the second for the maximal orthogonal family, and the third for the cardinal-characteristic bookkeeping. The third reservoir is included only to keep the side forcing separated from the two coding constructions; its stages do not themselves write stationary-set codes. For the two coding reservoirs we also fix recursive injections

$$\iota_\Sigma,\ \iota_{\mathsf{MO}} : 2^\omega \longrightarrow 2^\omega$$

with disjoint closed ranges. If $t \in 2^\omega$, put

$$\Delta(t) = \{2n+2 : n \in t\} \cup \{2n+1 : n \notin t\},$$

and set

$$\Delta_\Sigma(z) = \Delta(\iota_\Sigma(z)), \qquad \Delta_{\mathsf{MO}}(z) = \Delta(\iota_{\mathsf{MO}}(z)).$$

Thus a pattern on a block determines both the recovered real and the tag under which it was written. In particular, a real written for the maximal-orthogonal-family construction cannot be read as a global-uniformization certificate, and conversely.

Let $\tau \in \{\Sigma, \mathsf{MO}\}$ and let $\lambda \in \Lambda_\tau$ be fresh. In a $\mathbb{P}_\alpha$-extension, given a real z, the local coding forcing for writing z into I_λ is

$$\mathrm{Code}(\lambda, \tau, z) = \Big\{(s, F) : s \in [\omega]^{<\omega},\ F \in \big[\bigcup_{m \in \Delta_\tau(z)} Y_{\lambda+m} \times \{m\}\big]^{<\omega}\Big\}.$$

The order is the usual almost-disjoint coding order:

$$(t, F') \leq (s, F)$$

iff s is an initial segment of t, $F \subseteq F'$, and

$$(t \setminus s) \cap B_{\zeta,m} = \emptyset \qquad \text{for every } (\zeta, m) \in F,$$

where $\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1,\, m < \omega \rangle$ is the fixed almost disjoint family from Section 4. The generic real is

$$u_\lambda = \bigcup \{ s : (\exists F)\big((s, F) \in G_\lambda\big) \} \subseteq \omega.$$

For every $m \in \Delta_\tau(z)$ it almost-disjointly codes the localized set $Y_{\lambda+m}$:

$$\zeta \in Y_{\lambda+m} \quad \Longleftrightarrow \quad |u_\lambda \cap B_{\zeta,m}| < \omega.$$

For $m \notin \Delta_\tau(z)$ the coordinate $\lambda + m$ is not activated by the coding forcing. We put it into the protected set

$$A_\lambda = I_\lambda \setminus \{\lambda + m : m \in \Delta_\tau(z)\}.$$

The no-unwanted-codes lemma then says that if $\xi \in A_\lambda$, no real in the final extension codes a club in ω_2 disjoint from S^ξ. Hence the final model can distinguish activated coordinates from protected coordinates by a projective test.

The corresponding one-variable Fischer–Friedman–Zdomskyy coding predicate for the reservoir τ has the following form. Write $\Phi^\tau_{\mathrm{FFZ}}(z)$ for the assertion that there are a real u and a block $\lambda \in \Lambda_\tau$ such that, for every countable suitable transitive model M with $u, z \in M$, the model $(L[u])^M$ sees an internal block $\bar\lambda$ in the τ-reservoir with

$$(S^{\bar\lambda+m})^{(L[u])^M} \text{ nonstationary} \qquad \text{for all } m \in \Delta_\tau(z).$$

All quantification after u, z and M have been fixed is first-order over the coded model. Thus Φ^τ_{FFZ} is a Σ^1_3 predicate. Intentional coding gives the forward implication: if a stage writes z into a fresh block of Λ_τ, then $\Phi^\tau_{\mathrm{FFZ}}(z)$ holds in the final extension. The no-unwanted-codes lemma gives exactness in the direction needed later: any block which satisfies the projective decoding test must be one of the blocks intentionally activated by the construction, and all coordinates outside the associated pattern remain protected.

The forcing after P_0 is a finite-support iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_3 \rangle$$

of c.c.c., in fact σ-centered, iterands. At a stage using Λ_Σ or Λ_{MO} the iterand is one of the local forcings $\mathrm{Code}(\lambda, \tau, z)$, or the trivial forcing if the bookkeeping task is not successful. At a stage using $\Lambda_{\mathfrak{b}}$ the iterand is one of the prescribed side forcings for the bounding number or for adding Cohen reals, and its name is chosen with support below the next unused coding block. The bookkeeping is arranged so that each of the three kinds of stages occurs cofinally often after every initial segment, and so that every nontrivial coding stage uses a block which is fresh relative to all earlier stages and to the names currently being handled.

5.2 The maximal-orthogonal-family stages

At stages in Λ_{MO} the construction follows Fischer–Friedman–Törnquist. The bookkeeping presents a name $\dot{x}$ for a real. In the current intermediate model, if x is not a code for a non-atomic probability measure, or if μ_x is not orthogonal to the family already constructed, the stage is trivial.

If x codes a non-atomic measure orthogonal to the family constructed so far, the stage is successful. Choose a fresh block I_λ with $\lambda \in \Lambda_{\mathsf{MO}}$. The iterand is

$$\mathrm{Code}(\lambda, \mathsf{MO}, x).$$

It adds a real u_λ which writes the MO-tagged pattern of x into the block I_λ through the Fischer–Friedman–Zdomskyy almost-disjoint/stationary-set apparatus. In the next model the construction puts

$$g_\lambda = G(x, u_\lambda) \qquad \text{and} \qquad \mu_{g_\lambda} \in O.$$

By Lemma 3.1, $\mu_{g_\lambda} \approx \mu_x$ and g_λ FT-decodes to u_λ.

We spell out this decoding assertion, since this is the point at which the local Fischer–Friedman–Zdomskyy coding enters the projective definition of O. Write r_{FT} for the decoding map called r in Section 3. Let $\rho = g_\lambda$ and let M be a countable suitable transitive model with $\rho \in M$. The Fischer–Törnquist coding relation R and the map G are recursive. Hence M computes from ρ the same real

$$u = r_{\mathrm{FT}}(\rho) = u_\lambda,$$

and equivalently M satisfies $R(\rho, u)$. Once u is recovered, the statement $\rho = G(x, u)$ is arithmetical in the pair (x, u).

The remaining task is to recover x from the local stationary-set pattern written by u. In M we use the following first-order decoding relation over the coded model $(L[u])^M$. Say that *u locally* MO*-decodes to x in M* if there is an internal block start $\bar\lambda \in \Lambda^M_{\mathsf{MO}}$ such that, for every $m < \omega$,

$$m \in \Delta_{\mathsf{MO}}(x) \quad \Longleftrightarrow \quad (L[u])^M \models S^{\bar\lambda+m} \text{ is nonstationary.}$$

Equivalently, the right-hand side says that $(L[u])^M$ contains a club

$$C_m \subseteq (\omega_2)^M \qquad \text{with} \qquad C_m \cap (S^{\bar\lambda+m})^M = \emptyset.$$

All quantifiers in this displayed relation are first-order quantifiers over $(L[u])^M$, once the real parameters u, x and the countable model M have been fixed.

For the intentionally inserted real u_λ this local decoding relation holds. If $m \in \Delta_{\mathsf{MO}}(x)$, then the forcing $\mathrm{Code}(\lambda, \mathsf{MO}, x)$ makes u_λ almost-disjointly code $Y_{\lambda+m}$. Thus any countable suitable transitive M containing u_λ can recover, inside M, the initial segment $Y_{\lambda+m} \cap \omega_1^M$. By the localization property of the preliminary forcing, this initial segment yields in M the corresponding initial segments of $X_{\lambda+m}$ and X_λ, and the formula ϕ from Section 4 reconstructs the internal stationary kill of the M-copy of $S^{\lambda+m}$. This gives the forward implication in the displayed equivalence.

Conversely, if $m \notin \Delta_{\mathsf{MO}}(x)$, then $\lambda + m$ was placed into the protected part

$$A_\lambda = I_\lambda \setminus \{\lambda + k : k \in \Delta_{\mathsf{MO}}(x)\}.$$

The no-unwanted-codes lemma for the Fischer–Friedman–Zdomskyy blocks implies that no real in the final extension codes a club through the complement of $S^{\lambda+m}$. If $(L[u_\lambda])^M$ contained such a club for the corresponding internal coordinate, then, since M is countable in the ambient universe, this club would have a real code in the final extension. That would contradict the protected-coordinate clause. Hence no such club appears in $(L[u_\lambda])^M$, and the reverse implication follows. Thus every countable suitable transitive model containing g_λ verifies the local record

$$\exists x\, \exists u\, \big(R(g_\lambda, u) \wedge u \text{ locally } \mathsf{MO}\text{-decodes to } x \wedge g_\lambda = G(x, u)\big),$$

and the real x recovered in this way is the original measure code used at the successful stage.

Lemma 5.1. *In the final extension, the non-atomic part O of the maximal orthogonal family is exactly captured by a Π^1_2 condition. More precisely, for $\rho \in p_c(2^\omega)$,*

$$\mu_\rho \in O$$

if and only if for every countable suitable transitive model M with $\rho \in M$, M sees a successful maximal-orthogonal-family block for ρ: internally, ρ FT-decodes to a real u, the real u locally MO*-decodes to a non-atomic measure code x, and*

$$\rho = G(x, u).$$

Consequently O is Π^1_2-definable. Moreover, O is maximal among non-atomic measures, and adjoining the point masses gives a Π^1_2-definable maximal orthogonal family in $P(2^\omega)$.

Proof. We identify O with its set of codes when discussing projective complexity. Once the countable suitable transitive model M has been fixed, all clauses in the displayed condition are first-order, respectively arithmetical, over the coded model. The relation $R(\rho, u)$ expressing that ρ FT-decodes to u is arithmetical, the equation $\rho = G(x, u)$ is arithmetical because G is recursive, and the assertion that u locally MO-decodes to x is first-order over $(L[u])^M$: it says that there is an internal block start $\bar\lambda \in \Lambda^M_{\mathsf{MO}}$ such that, for every $m < \omega$,

$$m \in \Delta_{\mathsf{MO}}(x) \quad \Longleftrightarrow \quad (L[u])^M \models S^{\bar\lambda+m} \text{ is nonstationary.}$$

Equivalently, the right-hand side asserts the existence in $(L[u])^M$ of a club in $(\omega_2)^M$ disjoint from $(S^{\bar\lambda+m})^M$. Thus the only outer quantifier is the universal quantifier over real codes for countable suitable transitive models containing ρ, and the definition is Π^1_2.

Every intentionally inserted measure satisfies this condition. Suppose that at a successful MO-stage with block start λ the construction considered the non-atomic measure code x, forced with $\mathrm{Code}(\lambda, \mathsf{MO}, x)$, added the coding real u_λ, and inserted

$$\mu_{g_\lambda} \qquad \text{where} \qquad g_\lambda = G(x, u_\lambda).$$

Let M be a countable suitable transitive model containing g_λ. Since the FT-coding relation and G are recursive, M computes the same real $u_\lambda = r_{\mathrm{FT}}(g_\lambda)$ and verifies $R(g_\lambda, u_\lambda)$ and $g_\lambda = G(x, u_\lambda)$. The preceding local-decoding argument shows that, in M, the real u_λ locally MO-decodes to the original code x. Hence g_λ satisfies the Π^1_2 condition.

Conversely, suppose that a real $\rho \in p_c(2^\omega)$ satisfies the displayed condition. Then every countable suitable transitive model containing ρ sees a block start $\bar\lambda$ and a real u such that $R(\rho, u)$, u locally MO-decodes to a non-atomic code x, and $\rho = G(x, u)$. By the no-unwanted-codes lemma for the Fischer–Friedman–Zdomskyy blocks, the stationary-set pattern seen in such a block cannot occur at a protected coordinate. Therefore the block was intentionally used at a successful maximal-orthogonal-family stage. At that stage the construction inserted precisely the canonical code $G(x, u)$, and hence $\mu_\rho \in O$. This proves exactness of the Π^1_2 definition.

Orthogonality follows from the success criterion at the construction stages. At a successful stage the current measure is required to be orthogonal to all earlier inserted measures, and the inserted measure is equivalent to the current one by Lemma 3.1. Thus the family O is orthogonal. Maximality among non-atomic measures follows from the cofinal bookkeeping: every name for a non-atomic measure is eventually considered, and if it is still orthogonal to the previous family at that point, an equivalent measure is inserted. Finally, adjoining all point masses gives a maximal orthogonal family in $P(2^\omega)$, and the set of point masses is Borel, so the resulting family remains Π^1_2-definable. □

5.3 The global Σ-uniformization stages

At stages in Λ_Σ we run the copying construction for global Σ-uniformization. The base predicate is the Fischer–Friedman–Zdomskyy coding predicate $\Phi^\Sigma_{\mathrm{FFZ}}$ from the previous subsection: a real is declared to be coded only when it is written into a fresh Σ-tagged block by a forcing of the form

$$\mathrm{Code}(\lambda, \Sigma, z).$$

Thus the role played by the branch-versus-specialize predicate in the author's large-continuum uniformization construction is now played by the local Fischer–Friedman–Zdomskyy coding predicate attached to the Σ-reservoir.

We fix a recursive coding of finite tuples of reals by reals. If t is such a tuple, write $\ulcorner t \urcorner$ for its real code, and put

$$\mathrm{Coded}_\Sigma(t) \iff \Phi^\Sigma_{\mathrm{FFZ}}(\ulcorner t \urcorner).$$

We write $\mathrm{NotCoded}_\Sigma(t)$ for the negation of this predicate. Since $\Phi^\Sigma_{\mathrm{FFZ}}$ is Σ^1_3, Coded_Σ is Σ^1_3 and $\mathrm{NotCoded}_\Sigma$ is Π^1_3.

The bookkeeping uses the canonical ordering of nice names from the finite-support iteration. Names from shorter initial segments precede names which genuinely need longer initial segments, and ties are broken by the canonical wellorder of L. For a real x present at an intermediate stage, we list the distinct pairs of canonical names for reals as

$$(x, y_\xi, a_0^\xi)_{\xi<\omega_3},$$

with the convention that the first candidate is

$$(x, y_0, a_0^0) = (x, 0, 0).$$

This list is a bookkeeping list of names, not a reference to a final projective wellorder. The convention about the zeroth candidate only treats the harmless case in which the value 0 is already witnessed by the leading witness $a_0 = 0$; when this happens the uniformizing function is defined to take value 0.

We next spell out the compression device. This is the point at which we use the same trick as in the Martin's Axiom/global-Σ construction: the compression maps are chosen externally on canonical names, and their values are never changed later. Thus coherence is obtained at the level of names, instead of requiring the intermediate universes themselves to carry a canonical family of bijections.

Lemma 5.2. *There is an external family*

$$\langle \pi_\xi : 0 < \xi < \omega_3 \rangle$$

with the following properties.

(i) If $\dot b$ is a canonical name for a real and $0 < \xi < \omega_3$, then $\pi_\xi(\dot b)$ is a canonical name for a ξ-sequence of reals. We write

$$\pi_\xi(\dot b) = \langle \dot b^\eta : \eta < \xi \rangle.$$

(ii) Every canonical name for a ξ-sequence of reals occurs as $\pi_\xi(\dot b)$ for some canonical real name $\dot b$.

(iii) If $\dot b$ and the sequence $\langle \dot b^\eta : \eta < \xi \rangle$ are already names over $\mathbb{P}_\alpha$, then the same equality is used at every later stage. Hence, for every $\beta \geq \alpha$ and every $\mathbb{P}_\beta$-generic filter G_β,

$$\pi_\xi(\dot b)^{G_\beta} = \langle (\dot b^\eta)^{G_\beta} : \eta < \xi \rangle.$$

The maps π_ξ are bookkeeping devices only; they do not occur in the final projective definitions.

Proof. Let $\dot{\mathbb{P}}_{\omega_3}$ denote the later finite-support iteration after the preliminary forcing P_0. A nice name for a real uses countably many conditions, and every condition has finite support in the later iteration. Hence a name for a ξ-sequence of reals, where $\xi < \omega_3$, uses the union of at most $|\xi| \cdot \omega \leq \omega_2$ many finite supports. Since ω_3 is regular, this union is bounded in ω_3. Thus every relevant ξ-sequence name is already a name over some proper initial segment $\mathbb{P}_\alpha$.

The full forcing has size ω_3 and the ω_3-chain condition, and in L we have GCH. Therefore, for each fixed $\xi < \omega_3$, there are only ω_3 many canonical names for ξ-sequences of reals and only ω_3 many canonical names for reals. Choose in L, once and for all, a surjection from the latter class of names onto the former class of names, with the additional requirement that if a sequence name is supported by $\mathbb{P}_\alpha$, then some real name supported by a sufficiently large $\mathbb{P}_\beta$, $\beta < \omega_3$, is assigned to it. This is possible by the preceding boundedness paragraph and the cofinal repetition of names in the bookkeeping. Having made this assignment, we never redefine it. This gives (i)–(iii). □

We now define the action at a Σ-stage. Suppose the bookkeeping at stage α presents

$$(e, \dot{x}, \xi, \dot{b}_1, \dot{b}_2, \ldots),$$

where e is a Gödel number for a projective formula in two free variables, $\dot{x}$ and the $\dot{b}_i$ are names for reals, and $0 < \xi < \omega_3$ is a candidate index. Only finitely many of the reals b_i will be used, depending on the projective level of the formula. We evaluate the names in the current extension and write

$$x = (\dot{x})^{G_\alpha}, \qquad b_i = (\dot{b}_i)^{G_\alpha}.$$

We also use the name-level decoding

$$\pi_\xi(\dot{b}_i) = \langle \dot{b}_i^\eta : \eta < \xi \rangle, \qquad b_i^\eta = (\dot{b}_i^\eta)^{G_\alpha},$$

provided all these decoded names are supported by the current stage. If not, the bookkeeping request is ignored at this occurrence and will appear again later. When a nontrivial action is taken, we choose a fresh block I_λ with $\lambda \in \Lambda_\Sigma$, above the preliminary support of all names involved.

Odd projective levels. Assume that e codes a formula

$$\theta_e(x, y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y, a_0, a_1, \ldots, a_{2m-2}),$$

where $m \geq 1$ and ψ_e is Π^1_2. Let z^O_ξ be the real code of the finite tuple

$$(\mathsf{U}_{\mathrm{odd}}, e, x, y_\xi, a_0^\xi, b_1, \ldots, b_{2m-2}).$$

The stage performs the following test:

$$\forall \eta < \xi \; \neg\psi_e(x, y_\eta, a_0^\eta, b_1^\eta, \ldots, b_{2m-2}^\eta). \qquad (O_\xi)$$

If (O_ξ) holds, then

$$\dot{\mathbb{Q}}_\alpha = \mathrm{Code}(\lambda, \Sigma, z^O_\xi).$$

If (O_ξ) fails, then $\dot{\mathbb{Q}}_\alpha$ is trivial. Thus, on odd levels, compressed evidence that all earlier candidates fail is copied by positive coding. If some earlier candidate succeeds, the construction records this by not intentionally coding the corresponding tuple. In the final model the negative information is read from the no-unwanted-codes theorem: a tuple is Σ-coded only if it was intentionally written into a fresh Σ-block.

Even projective levels. Assume that e codes a formula

$$\theta_e(x, y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \forall a_{2m-3} \psi_e(x, y, a_0, a_1, \ldots, a_{2m-3}),$$

where $m \geq 2$ and ψ_e is Σ^1_2. Let z^E_ξ be the real code of

$$(\mathsf{U}_{\mathrm{even}}, e, x, y_\xi, a_0^\xi, b_1, \ldots, b_{2m-3}).$$

The stage tests

$$\forall \eta < \xi \; \neg\psi_e(x, y_\eta, a_0^\eta, b_1^\eta, \ldots, b_{2m-3}^\eta). \qquad (E_\xi)$$

If (E_ξ) holds, the stage is trivial. If (E_ξ) fails, then

$$\dot{\mathbb{Q}}_\alpha = \mathrm{Code}(\lambda, \Sigma, z^E_\xi).$$

This is the exact dual of the odd rule: at even levels failure of the compressed test is copied by positive coding, while success of the compressed test is recorded by the absence of intentional coding.

The following lemma records the bookkeeping completeness needed in the proof. We use Shoenfield absoluteness in the standard form that Σ^1_2 and Π^1_2 statements with real parameters already present at a stage have the same truth value in that intermediate extension and in any later forcing extension produced by the construction.

Lemma 5.3. *Let e, x, $0 < \xi < \omega_3$, and finitely many reals $b_1, \ldots, b_k$ belong to the final extension. Suppose that the canonical names for the candidates $(x, y_\eta, a^\eta_0)_{\eta \le \xi}$ and the decoded sequence names*

$$\pi_\xi(\dot{b}_i) = \langle \dot{b}^\eta_i : \eta < \xi \rangle \qquad (1 \le i \le k)$$

are fixed. Then there is a Σ-stage after all these names are supported at which the bookkeeping presents precisely the request

$$(e, \dot{x}, \xi, \dot{b}_1, \ldots, \dot{b}_k).$$

At that stage the truth value of the relevant displayed Π^1_2 or Σ^1_2 matrix statements agrees with their truth value in the final model. Consequently the construction either intentionally codes the corresponding tuple or leaves the stage trivial according to the odd/even rule above. In the trivial case, every later occurrence of the same supported canonical tuple has the same matrix truth value, and therefore no later occurrence intentionally codes it.

Proof. The boundedness argument from Lemma 5.2 shows that the real parameters, the candidate names, and the decoded sequence names all appear in some proper initial segment of the finite-support iteration. The Σ-stage bookkeeping repeats every finite request cofinally often after the relevant support has appeared, so a later stage presents exactly this request. The matrix statements being evaluated are Π^1_2 or Σ^1_2 with real parameters in the intermediate model. Shoenfield absoluteness gives the same truth value in the final extension. The last assertion follows directly from the definition of the stage and Shoenfield absoluteness. □

Lemma 5.4. *Work in the final model.*

(1) Suppose

$$\theta_e(x, y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y, a_0, \ldots, a_{2m-2})$$

is an odd-level formula with $\psi_e \in \Pi^1_2$, and suppose ξ is successful, that is

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y_\xi, a^\xi_0, a_1, \ldots, a_{2m-2})$$

holds. Then for every $\beta > \xi$,

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \ \mathrm{NotCoded}_\Sigma(\mathsf{U}_{\mathrm{odd}}, e, x, y_\beta, a^\beta_0, a_1, \ldots, a_{2m-2}). \tag{1}$$

(2) Suppose the formula is on an even level,

$$\theta_e(x, y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \forall a_{2m-3} \psi_e(x, y, a_0, \ldots, a_{2m-3}),$$

with $\psi_e \in \Sigma^1_2$, and suppose ξ is successful. Then for every $\beta > \xi$,

$$\forall a_1 \exists a_2 \cdots \forall a_{2m-3} \ \mathrm{Coded}_\Sigma(\mathsf{U}_{\mathrm{even}}, e, x, y_\beta, a^\beta_0, a_1, \ldots, a_{2m-3}). \tag{2}$$

Proof. We prove the odd case; the even case is obtained by reversing the coding and trivial-stage clauses. Fix $\beta > \xi$ and let a_1 be arbitrary. Decode a_1 by the coherent compression map:

$$\pi_\beta(a_1) = \langle a_1^\eta : \eta < \beta \rangle.$$

Since the candidate ξ is successful, there are witnesses $a_2^\xi, \dots, a_{2m-2}^\xi$ such that

$$\psi_e(x, y_\xi, a_0^\xi, a_1^\xi, a_2^\xi, \dots, a_{2m-2}^\xi)$$

holds. Choose arbitrary values for the other coordinates, and compress the resulting families into reals $a_2, \dots, a_{2m-2}$ by the same coherent maps. By Lemma 5.3, the request for

$$(\mathsf{U}_{\text{odd}}, e, x, y_\beta, a_0^\beta, a_1, \dots, a_{2m-2})$$

is eventually presented at a Σ-stage after all relevant names are supported. At that stage the test (O_β) fails, since the coordinate $\eta = \xi$ makes the Π^1_2 matrix true. The stage is therefore trivial. Any earlier occurrence of the same canonical tuple was either ignored because the necessary decoded names were not yet supported, or had the same matrix truth value and was also trivial; the same is true at all later occurrences by Shoenfield absoluteness. Hence the tuple is never intentionally coded. By the no-unwanted-codes theorem for the Σ-reservoir, a real satisfying $\Phi^\Sigma_{\mathrm{FFZ}}$ must have been intentionally coded in some fresh Σ-block. Thus the tuple is not coded in the final model. This proves (1).

For even levels, the same argument shows that the test (E_β) fails. The even rule then deliberately codes the corresponding tuple, and the intentional coding implication for $\Phi^\Sigma_{\mathrm{FFZ}}$ gives (2). □

Lemma 5.5. *Work in the final model.*

(1) In the odd-level case, suppose $\xi > 0$ is least such that

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y_\xi, a_0^\xi, a_1, \dots, a_{2m-2})$$

holds. Then

$$\exists a_1 \forall a_2 \cdots \forall a_{2m-2}\ \mathrm{Coded}_\Sigma(\mathsf{U}_{\text{odd}}, e, x, y_\xi, a_0^\xi, a_1, \dots, a_{2m-2}). \tag{3}$$

(2) In the even-level case, if $\xi > 0$ is least successful, then

$$\exists a_1 \forall a_2 \cdots \exists a_{2m-3}\ \mathrm{NotCoded}_\Sigma(\mathsf{U}_{\text{even}}, e, x, y_\xi, a_0^\xi, a_1, \dots, a_{2m-3}). \tag{4}$$

Proof. Again we give the odd proof. Since ξ is the least successful candidate, for each $\eta < \xi$ the tail assertion for (x, y_η, a_0^η) fails. Thus for each $\eta < \xi$ there are counter-witnesses arranged according to the dual quantifier pattern. Compress the first layer of these counter-witnesses into a single real a_1 using π_ξ. Now let a_2 be arbitrary and decode it as a ξ-sequence. For each $\eta < \xi$, choose the next layer of counter-witnesses; compress that layer into a_3, and continue through the finite alternating pattern. The result is that for every play of the universal variables in (3), all coordinates $\eta < \xi$ make

$$\psi_e(x, y_\eta, a_0^\eta, \dots)$$

false.

For the corresponding tuple, Lemma 5.3 gives a sufficiently late Σ-stage at which the request is presented. At that stage (O_ξ) holds. Therefore the construction forces with

$\mathrm{Code}(\lambda, \Sigma, z)$ for the real code of the tuple. The intentional-coding implication for $\Phi^{\Sigma}_{\mathrm{FFZ}}$ proves (3).

In the even case the same compressed counter-witnesses make (E_ξ) hold. The stage is therefore trivial. Any occurrence of the same supported canonical tuple has the same matrix truth value by Shoenfield absoluteness, and so no occurrence intentionally codes it. The no-unwanted-codes theorem for the Σ-reservoir therefore gives the displayed $\mathrm{NotCoded}_\Sigma$ statement. □

Lemma 5.6. *In the final extension of the large-continuum construction, every Σ^1_n set of pairs of reals, for $n \geq 2$, has a Σ^1_n uniformization.*

Proof. The case $n = 2$ is Kondo's theorem. We handle the odd levels; the even levels are dual. Let

$$\theta_e(x, y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y, a_0, a_1, \ldots, a_{2m-2}), \qquad \psi_e \in \Pi^1_2,$$

be a Σ^1_{2m+1} formula. Define $U_e(x, y)$ as follows. First, $U_e(x, 0)$ holds if

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, 0, 0, a_1, \ldots, a_{2m-2})$$

holds. If this initial clause fails, then $U_e(x, y)$ holds iff there is a real a_0 such that

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y, a_0, a_1, \ldots, a_{2m-2})$$

and

$$\neg\Big[\forall a_1 \exists a_2 \cdots \exists a_{2m-2}\ \mathrm{NotCoded}_\Sigma(\mathsf{U}_{\mathrm{odd}}, e, x, y, a_0, a_1, \ldots, a_{2m-2})\Big].$$

This is a Σ^1_{2m+1} definition: the last displayed negation is equivalent at the required complexity to the existence of a play of the dual variables for which the base Σ^1_3 predicate Coded_Σ holds.

Suppose the x-section of θ_e is nonempty. If the initial candidate $(x, 0, 0)$ is successful, the first clause assigns value 0. Otherwise let $\xi > 0$ be least such that the candidate (x, y_ξ, a^ξ_0) is successful. Lemma 5.5 gives the coded witness required by the second clause, so $U_e(x, y_\xi)$ holds. If $\beta < \xi$, the original tail matrix for (x, y_β, a^β_0) fails by minimality of ξ. If $\beta > \xi$, Lemma 5.4 gives the full $\mathrm{NotCoded}_\Sigma$ tail, so the negated bracket in the definition fails. Hence exactly one y is selected, and it satisfies $\theta_e(x, y)$.

For even levels, with

$$\theta_e(x, y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \forall a_{2m-3} \psi_e(x, y, a_0, a_1, \ldots, a_{2m-3}), \qquad \psi_e \in \Sigma^1_2,$$

the same definition is used with the parity reversed: replace the last displayed $\mathrm{NotCoded}_\Sigma$ by Coded_Σ and use Lemma 5.5(2) together with Lemma 5.4(2). The resulting predicate is Σ^1_{2m} and uniformizes the given Σ^1_{2m} relation. □

5.4 The cardinal-characteristic stages

At stages in $\Lambda_{\mathfrak{b}}$ we take care of the cardinal characteristics needed for Theorem 1.1. Cofinally often we add dominating reals, using the standard σ-centered dominating forcing. Cofinally often we also add Cohen reals. These stages are placed on blocks disjoint from both Λ_Σ and Λ_{MO}.

The finite-support c.c.c. iteration has length ω_3 and all iterands have size at most ω_2 in the relevant intermediate models. Hence the final continuum is at most ω_3, and the cofinally many real-adding stages give $\mathfrak{c} \geq \omega_3$. Thus $\mathfrak{c} = \omega_3$.

The cofinally many dominating stages give $\mathfrak{b} = \omega_3$. Indeed, any family of fewer than ω_3 many reals appears in some intermediate model, and a later dominating stage adds a real eventually dominating all reals in that intermediate model. Therefore no family of size $< \omega_3$ is unbounded. Since always $\mathfrak{b} \leq \mathfrak{c}$, we obtain

$$\mathfrak{b} = \mathfrak{c} = \omega_3.$$

The cofinally many Cohen stages ensure that for every real parameter a in the final extension there is a Cohen real over $L[a]$. Proposition 2.3 then gives that there are no boldface Σ^1_2 maximal orthogonal families.

6 Proof of the first main theorem

Theorem 6.1. *Assume $V = L$. Let*

$$\mathbb{P} = P_0 * \dot{\mathbb{P}}_{\omega_3}$$

be the finite-support forcing constructed in Section 5. If $G \subseteq \mathbb{P}$ is generic over L, then in $L[G]$,

$$\mathfrak{b} = \mathfrak{c} = \omega_3,$$

there is a Π^1_2 maximal orthogonal family of Borel probability measures on 2^ω, there are no boldface Σ^1_2 maximal orthogonal families, and UP_Σ holds.

Proof. Write $G = G_0 * H$, where $G_0 \subseteq P_0$ is generic for the preliminary Fischer–Friedman–Zdomskyy forcing and $H \subseteq \mathbb{P}^{G_0}_{\omega_3}$ is generic for the finite-support tail. By Lemma 4.1, P_0 is ω-distributive. In particular it adds no reals. The preservation facts recorded in Section 4 give that P_0 preserves cardinals. The tail forcing is a finite-support iteration of c.c.c., indeed σ-centered, iterands of length ω_3. Hence the full forcing preserves cardinals.

The continuum in $L[G]$ is ω_3. The preliminary forcing adds no reals. The tail has length ω_3 and all real names are supported on countably many tail coordinates together with a preliminary support below some block; consequently there are at most ω_3 many reals in the final extension. Conversely, the bookkeeping contains cofinally many real-adding stages, and in particular cofinally many Cohen and dominating stages, so at least ω_3 many reals are added. Thus

$$\mathfrak{c}^{L[G]} = \omega_3.$$

The same cardinal-characteristic bookkeeping gives $\mathfrak{b} = \omega_3$. Let $F \subseteq \omega^\omega$ have size $< \omega_3$ in $L[G]$. Since the tail is a finite-support c.c.c. iteration of regular length ω_3, all names for members of F appear in some intermediate model $L[G_0][H \restriction \alpha]$ for some $\alpha < \omega_3$. Choose a later stage in $\Lambda_\mathfrak{b}$ at which the construction adds a dominating real over that intermediate model. This real eventually dominates every member of F. Hence no family of size $< \omega_3$ is unbounded in $L[G]$, so $\mathfrak{b} \geq \omega_3$. Since $\mathfrak{b} \leq \mathfrak{c} = \omega_3$, we get

$$\mathfrak{b} = \mathfrak{c} = \omega_3.$$

The maximal orthogonal family is obtained from the stages using the reservoir Λ_{MO}. Let O be the non-atomic part produced there. Lemma 5.1 says that O is exactly defined by a Π^1_2 condition, that its elements are pairwise orthogonal, and that it is maximal among non-atomic measures. The family

$$\mathcal{A} = O \cup \{\delta_x : x \in 2^\omega\}$$

obtained by adjoining the point masses is therefore a maximal orthogonal family in $\mathcal{P}(2^\omega)$. The set of point masses is Borel, so $\mathcal{A}$ is still Π^1_2-definable.

There are no boldface Σ^1_2 maximal orthogonal families in $L[G]$. Let a be any real in the final extension. Then a belongs to some intermediate model $L[G_0][H \restriction \alpha]$. By the cofinal

Cohen bookkeeping, there is a later Cohen stage. The Cohen real added there is Cohen over the intermediate model and hence Cohen over $L[a]$. Proposition 2.3 applies and rules out a $\Sigma^1_2(a)$ maximal orthogonal family. Since a was arbitrary, there are no boldface $\mathbf{\Sigma}^1_2$ maximal orthogonal families.

It remains to record the uniformization property. The stages using the reservoir Λ_Σ are governed by the global Σ-bookkeeping of Subsection 5.3. Lemma 5.2 supplies the coherent name-level compression maps used to compare all earlier candidates with a given candidate. Lemma 5.3 ensures that every relevant finite tuple of names is eventually presented to the construction. Lemma 5.4 shows that every candidate which comes after the least successful one receives the appropriate exclusion pattern, and Lemma 5.5 shows that the least successful candidate receives the selecting pattern. Lemma 5.6 then gives that every Σ^1_n set of pairs of reals, for $n \geq 2$, has a Σ^1_n uniformization in $L[G]$. Thus $L[G] \models \mathsf{UP}_\Sigma$.

Finally, the two projective constructions do not interfere. The maximal-orthogonal-family stages use only blocks from Λ_{MO} and the tag MO, while the uniformization stages use only blocks from Λ_Σ and the Σ-tags. These reservoirs are disjoint, and the recursive injections ι_{MO} and ι_Σ have disjoint ranges. The no-unwanted-codes theorem for the Fischer–Friedman–Zdomskyy blocks therefore prevents a code written for one purpose from being decoded as a code for the other. This is exactly the separation needed to superpose the Fischer–Friedman–Törnquist maximal-family construction with the global Σ-uniformization construction. □

Theorem 1.1 is precisely Theorem 6.1.

7 The second model

We now prove Theorem 1.2. The proof is deliberately shorter than the proof of Theorem 1.1. The point is that the projective analysis from Section 5 uses only an abstract coding package:

$$\text{intentional coding} + \text{no accidental coding} + \text{coherent copying}.$$

For the second model we replace the finite-support Fischer–Friedman–Zdomskyy blocks by the countable-support S-proper coding blocks used in the second Fischer–Friedman–Törnquist construction. The maximal-orthogonal-family stages and the global Σ-uniformization stages are then the same modules as before.

7.1 The ω_1-coding package

Fix in L a uniformly definable almost disjoint sequence

$$\vec{S} = \langle S_\xi : \xi < \omega_2 \rangle$$

of stationary subsets of ω_1, and fix a stationary set

$$S \subseteq \omega_1$$

which is almost disjoint from every S_ξ. For a stationary co-stationary set $T \subseteq \omega_1$, let

$$Q(T) = \{c : c \subseteq \omega_1 \setminus T \text{ is countable closed}\},$$

ordered by end-extension. Thus $Q(T)$ adds a club subset of $\omega_1 \setminus T$. If $T = S_\xi$, then $Q(T)$ is S-proper, because $S \cap S_\xi$ is nonstationary. It is also ω-distributive, hence ω^ω-bounding.

In this section a transitive model M is called ω_2*-suitable* if ω_2^M exists and

$$\omega_2^M = (\omega_2)^{L^M}.$$

Then also $\omega_1^M = (\omega_1)^{L^M}$. The projective definitions in this section quantify over real codes for countable ω_2-suitable transitive models.

A coding block is now an interval

$$I_\lambda = \{\lambda + m : m < \omega\}, \qquad 0 < \lambda \in \mathrm{Lim}(\omega_2).$$

As before, the chosen block starts are partitioned into two disjoint cofinal reservoirs

$$\Lambda_{\mathsf{MO}}, \qquad \Lambda_\Sigma,$$

and we use the same recursive tags

$$\iota_{\mathsf{MO}}, \iota_\Sigma : 2^\omega \to 2^\omega, \qquad \Delta_\tau(z) = \Delta(\iota_\tau(z)) \quad (\tau \in \{\mathsf{MO}, \Sigma\}).$$

Here

$$\Delta(t) = \{2n+2 : n \in t\} \cup \{2n+1 : n \notin t\}.$$

Let $\lambda \in \Lambda_\tau$ be fresh and let $z \in 2^\omega$. The forcing

$$\mathrm{Code}_{\omega_1}(\lambda, \tau, z)$$

is the ω_1-version of the local coding forcing. It is a three-step forcing

$$K^0(\lambda, \tau, z) * \dot{K}^1(\lambda, \tau, z) * \dot{K}^2(\lambda, \tau, z).$$

The first factor is the countable-support product

$$K^0(\lambda, \tau, z) = \prod_{m \in \Delta_\tau(z)} Q(S_{\lambda+m}).$$

It kills exactly the stationary sets $S_{\lambda+m}$ with $m \in \Delta_\tau(z)$, by adding clubs

$$C_{\lambda+m} \subseteq \omega_1 \setminus S_{\lambda+m}.$$

The coordinates $\lambda + m$ with $m \notin \Delta_\tau(z)$ are protected.

In the extension by $K^0(\lambda, \tau, z)$, let

$$X_{\lambda,\tau,z} \subseteq \omega_1$$

be the canonical least set coding

$$\lambda, \tau, z, \qquad \langle C_{\lambda+m} : m \in \Delta_\tau(z) \rangle,$$

together with the finite amount of bookkeeping needed to recognize the block. The second factor is the standard localization forcing

$$K^1(\lambda, \tau, z) = L(\varphi_{\lambda,\tau,z}),$$

where $\varphi_{\lambda,\tau,z}$ says that the decoded set X carries a block whose killed coordinates are exactly $\Delta_\tau(z)$. This is the localization forcing used in the ω_2 Fischer–Friedman–Törnquist construction. It has a countably closed dense subset and gives the following local property: if $Y_{\lambda,\tau,z}$ is the generic localization and M is countable ω_2-suitable with

$$Y_{\lambda,\tau,z} \cap \omega_1^M \in M,$$

then M internally reconstructs the initial segment of the coding record and verifies the corresponding instance of $\varphi_{\lambda,\tau,z}$.

Finally $K^2(\lambda, \tau, z)$ is the perfect-tree coding

$$C(Y_{\lambda,\tau,z}).$$

It adds a real

$$r_{\lambda,\tau,z}$$

which codes $Y_{\lambda,\tau,z}$ in the local sense. The forcing $C(Y)$ is proper and ω^ω-bounding. Hence

$$\mathrm{Code}_{\omega_1}(\lambda, \tau, z)$$

is S-proper and ω^ω-bounding.

There is a corresponding one-variable predicate. Write

$$\mathrm{Coded}_\tau^{\omega_1}(z)$$

for the assertion that there is a real r and a block start $\lambda \in \Lambda_\tau$ such that for every countable ω_2-suitable transitive model M with $r, z \in M$, the model $(L[r])^M$ sees an internal τ-block $\bar{\lambda}$ satisfying

$$(L[r])^M \models S_{\bar{\lambda}+m} \text{ is nonstationary} \quad \Longleftrightarrow \quad m \in \Delta_\tau(z).$$

Equivalently, $(L[r])^M$ contains clubs

$$C_m \subseteq \omega_1^M \setminus (S_{\bar{\lambda}+m})^M \qquad (m \in \Delta_\tau(z))$$

and contains no such clubs for the protected coordinates. The predicate $\mathrm{Coded}_\tau^{\omega_1}$ is Σ_3^1: it has the form

$$\exists r\, \forall M\, \Theta(r, z, M),$$

where M ranges over real codes for countable ω_2-suitable transitive models and Θ is first-order over the coded model. We write

$$\mathrm{NotCoded}_\tau^{\omega_1}(z) \quad \Longleftrightarrow \quad \neg\, \mathrm{Coded}_\tau^{\omega_1}(z).$$

Lemma 7.1. *Let*

$$\mathbb{R} = \langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\alpha : \alpha < \delta \rangle$$

be a countable-support iteration such that, for every $\alpha < \delta$,

$$\Vdash_{\mathbb{R}_\alpha} \dot{\mathbb{S}}_\alpha \textit{ is either trivial or of the form } \mathrm{Code}_{\omega_1}(\lambda, \tau, \dot{z}),$$

where $\tau \in \{\mathsf{MO}, \Sigma\}$, $\lambda \in \Lambda_\tau$, and $\dot{z}$ is a name for a real. Suppose that no block is used twice. Then in the final extension by $\mathbb{R}$, for every $\tau \in \{\mathsf{MO}, \Sigma\}$, every $\lambda \in \Lambda_\tau$, and every real z, the following are equivalent:

(i) for every $m < \omega$,

$$S_{\lambda+m} \textit{ is nonstationary} \quad \Longleftrightarrow \quad m \in \Delta_\tau(z);$$

(ii) at some stage the construction intentionally used the block I_λ and forced with

$$\mathrm{Code}_{\omega_1}(\lambda, \tau, z).$$

Consequently, if $\mathrm{Coded}_\tau^{\omega_1}(z)$ holds in the final model, then z was intentionally coded at a τ-stage. No protected coordinate can be turned into a code by later coding forcings.

Proof. The implication from (ii) to (i) is immediate from the definition of $K^0(\lambda, \tau, z)$. For the converse, fix a coordinate $\lambda + m$ which was not intentionally killed. Replace the sequence $\vec{S}$ by the sequence obtained by putting

$$\omega_1 \setminus S_{\lambda+m}$$

at this coordinate and leaving all other coordinates unchanged. Since $S_{\lambda+m}$ and its complement are stationary, and since all the other coding sets remain stationary, the modified sequence is still suitable for the usual S-proper preservation argument.

Each proper factor preserves this modified sequence. Each club-shooting factor which occurs in an intentional code also preserves it, unless it is exactly the forbidden factor $Q(S_{\lambda+m})$; but by assumption that factor never occurs. The standard S-proper preservation theorem for countable-support iterations therefore implies that $\omega_1 \setminus S_{\lambda+m}$ remains stationary throughout the iteration. Hence no club disjoint from $S_{\lambda+m}$ can appear in the final model.

Thus the only stationary sets killed by the iteration are those killed by intentional K^0 factors. The tag maps ι_{MO} and ι_Σ have disjoint ranges, so a killed pattern determines a unique tag and a unique real. The local real r added by the perfect-tree coding gives the projective witness for the intended code. This proves the lemma. ☐

7.2 Definition of the iteration

We now define the countable-support iteration which yields the second model. This subsection contains the bookkeeping for both the maximal-orthogonal-family construction and the global Σ-uniformization construction. Thus there is no separate transfer step: the copying machinery is built directly into the iteration.

We work over L. Let $\mathbb{B}$ denote the random algebra. We define a countable-support iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$$

by recursion. At successor stages we write

$$\dot{\mathbb{Q}}_\alpha = \mathbb{B} * \dot{\mathbb{R}}_\alpha.$$

Thus every stage first adds a random real. The second factor $\dot{\mathbb{R}}_\alpha$ is then determined by the bookkeeping and is either trivial or one of the ω_1-coding forcings

$$\mathrm{Code}_{\omega_1}(\lambda, \tau, \dot{z}), \qquad \tau \in \{\mathsf{MO}, \Sigma\}.$$

At limit stages we take countable supports.

Along the construction we maintain the non-atomic part of the orthogonal family,

$$\langle O_\alpha : \alpha \leq \omega_2 \rangle,$$

with $O_\delta = \bigcup_{\alpha<\delta} O_\alpha$ at limits.

We fix once and for all a recursive coding of finite tuples of reals by reals. If t is such a tuple, write $\ulcorner t \urcorner$ for its real code, and use the abbreviations

$$\mathrm{Coded}^{\omega_1}_\Sigma(t) \iff \mathrm{Coded}^{\omega_1}_\Sigma(\ulcorner t \urcorner),$$

and

$$\mathrm{NotCoded}^{\omega_1}_\Sigma(t) \iff \neg\, \mathrm{Coded}^{\omega_1}_\Sigma(t).$$

The same convention is used for the MO-tag when necessary.

The Σ-bookkeeping uses coherent name-level compression maps. For each $0 < \xi < \omega_2$ we fix externally a map

$$\pi_\xi : \{\text{canonical real names}\} \longrightarrow \{\text{canonical names for } \xi\text{-sequences of reals}\}.$$

If

$$\pi_\xi(\dot b) = \langle \dot b^\eta : \eta < \xi \rangle,$$

then this equality is never redefined at later stages. The maps are chosen so that every canonical name for a ξ-sequence of reals is the value of π_ξ on some canonical real name. Moreover, if the name $\langle \dot b^\eta : \eta < \xi \rangle$ is supported by an initial segment of the iteration, then some preimage $\dot b$ is chosen with support in a sufficiently large proper initial segment. This is possible because a name for a ξ-sequence of reals, with $\xi < \omega_2$, uses fewer than ω_2 many countable supports, and hence is supported below some stage $< \omega_2$.

For a real x appearing at an intermediate stage, the distinct canonical candidate triples are listed as

$$(x, y_\xi, a_0^\xi)_{\xi < \omega_2}.$$

The ordering is the canonical order of names: names from shorter initial segments come first, and ties are broken by the canonical wellorder of L. We use the convention

$$(x, y_0, a_0^0) = (x, 0, 0).$$

This is only a bookkeeping list of names; it is not used as a final projective wellorder.

MO-stages. The bookkeeping presents a canonical name $\dot x$ for a real over the current model after the initial random factor. In the $\mathbb{P}_\alpha * \mathbb{B}$-extension, if x is not a code for a non-atomic probability measure, or if μ_x is not orthogonal to the family constructed so far, then the second factor is trivial:

$$\dot{\mathbb{R}}_\alpha = \mathbf{1}.$$

If x codes a non-atomic measure orthogonal to O_α, choose a fresh block I_λ with $\lambda \in \Lambda_{\mathsf{MO}}$ and set

$$\dot{\mathbb{R}}_\alpha = \mathrm{Code}_{\omega_1}(\lambda, \mathsf{MO}, x).$$

Let $r_{\lambda,\mathsf{MO},x}$ be the real added by the perfect-tree coding part of this forcing. In the next model we put

$$g_\lambda = G(x, r_{\lambda,\mathsf{MO},x})$$

and add

$$\mu_{g_\lambda}$$

to the non-atomic orthogonal family. Thus

$$O_{\alpha+1} = O_\alpha \cup \{\mu_{g_\lambda}\}$$

at successful MO-stages, and $O_{\alpha+1} = O_\alpha$ otherwise.

Σ-stages. The bookkeeping presents a finite request

$$(e, \dot x, \xi, \dot b_1, \dots, \dot b_k), \qquad 0 < \xi < \omega_2,$$

where e is a Gödel number for a projective formula, $\dot x$ and the $\dot b_i$ are canonical names for reals, and k is the finite number of auxiliary reals required by the projective level of e. The request is read in the current $\mathbb{P}_\alpha * \mathbb{B}$-extension. If one of the decoded sequence names

$$\pi_\xi(\dot b_i) = \langle \dot b_i^\eta : \eta < \xi \rangle$$

is not yet supported by the current stage, the request is ignored at this occurrence and the second factor is trivial. The bookkeeping repeats the same finite request cofinally often, so it will reappear after the relevant supports have entered the iteration.

Assume first that e codes an odd-level formula

$$\theta_e(x,y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x,y,a_0,a_1,\dots,a_{2m-2}), \qquad \psi_e \in \Pi^1_2.$$

Let z^O_ξ be the real code of the tuple

$$(\mathsf{U}_{\mathrm{odd}}, e, x, y_\xi, a^\xi_0, b_1, \dots, b_{2m-2}).$$

The stage tests

$$\forall \eta < \xi \ \neg\psi_e(x, y_\eta, a^\eta_0, b^\eta_1, \dots, b^\eta_{2m-2}). \tag{O_ξ}$$

If (O_ξ) holds, choose a fresh block I_λ with $\lambda \in \Lambda_\Sigma$ and set

$$\dot{\mathbb{R}}_\alpha = \mathrm{Code}_{\omega_1}(\lambda, \Sigma, z^O_\xi).$$

If (O_ξ) fails, then the second factor is trivial. Thus, on odd levels, positive coding records that all earlier candidates have failed, while a successful earlier candidate is recorded by the absence of intentional coding.

Assume next that e codes an even-level formula

$$\theta_e(x,y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \forall a_{2m-3} \psi_e(x,y,a_0,a_1,\dots,a_{2m-3}), \qquad \psi_e \in \Sigma^1_2.$$

Let z^E_ξ be the real code of

$$(\mathsf{U}_{\mathrm{even}}, e, x, y_\xi, a^\xi_0, b_1, \dots, b_{2m-3}).$$

The stage tests

$$\forall \eta < \xi \ \neg\psi_e(x, y_\eta, a^\eta_0, b^\eta_1, \dots, b^\eta_{2m-3}). \tag{E_ξ}$$

If (E_ξ) fails, choose a fresh block I_λ with $\lambda \in \Lambda_\Sigma$ and set

$$\dot{\mathbb{R}}_\alpha = \mathrm{Code}_{\omega_1}(\lambda, \Sigma, z^E_\xi).$$

If (E_ξ) holds, the second factor is trivial.

Thus the odd and even rules are dual. On odd levels, the positive case of the compressed test is copied by coding and the negative case by omission. On even levels, the negative case is copied by coding and the positive case by omission. The no-unwanted-codes theorem turns this omission into the final projective NotCoded-information.

Trivial stages. If the bookkeeping request is malformed, or is not an MO-request or a Σ-request of the above forms, then

$$\dot{\mathbb{R}}_\alpha = \mathbf{1}.$$

The bookkeeping is chosen so that every real name, every measure name, every candidate triple, and every finite tuple of names needed for the Σ-copying construction is presented cofinally often. Whenever a nontrivial coding stage is performed, the block I_λ is chosen fresh: no coordinate in I_λ has been used earlier, protected earlier, or appears in the support of any of the names currently being acted on.

All second factors are S-proper and ω^ω-bounding. Indeed, the club-shooting factors appearing inside $\mathrm{Code}_{\omega_1}(\lambda, \tau, z)$ are S-proper and ω-distributive, the localization factors have countably closed dense subsets, and the perfect-tree coding factors are proper and ω^ω-bounding. Random forcing is proper, S-proper, and ω^ω-bounding. Hence the full countable-support iteration is S-proper and ω^ω-bounding by the standard preservation theorem for this class of iterations. We shall also use the standard random-preservation fact for this construction: the club-shooting and localization parts add no reals, the perfect-tree coding parts have the relevant Sacks-type preservation property, and the random algebra preserves random reals over earlier models.

7.3 Verification of the second theorem

We first record the Σ-stage analysis in the present context. This is the direct ω_1-coding version of the copying lemmas from Subsection 5.3; the point is that the statement below refers only to the predicates and bookkeeping of the present iteration.

Lemma 7.2. *Work in the final extension of the iteration from Subsection 7.2. The following hold.*

(i) *For every $0 < \xi < \omega_2$, the compression maps π_ξ are coherent: if*

$$\pi_\xi(\dot{b}) = \langle \dot{b}^\eta : \eta < \xi \rangle$$

is fixed at some stage, then the same equality is used at all later stages. Every canonical name for a ξ-sequence of reals is obtained in this way from some canonical real name.

(ii) *The bookkeeping is complete. Whenever e, x, $0 < \xi < \omega_2$, and finitely many reals $b_1, \ldots, b_k$ occur in the final extension, and whenever the candidate names*

$$(x, y_\eta, a_0^\eta)_{\eta \le \xi}$$

and the decoded sequence names

$$\pi_\xi(\dot{b}_i) = \langle \dot{b}_i^\eta : \eta < \xi \rangle \qquad (1 \le i \le k)$$

are fixed, there is a sufficiently late Σ-stage at which the construction presents exactly the request

$$(e, \dot{x}, \xi, \dot{b}_1, \ldots, \dot{b}_k).$$

At that stage the relevant Σ^1_2 or Π^1_2 matrix statements have the same truth value as in the final extension.

(iii) *Later candidates are copied as exclusions. Suppose*

$$\theta_e(x,y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y, a_0, \ldots, a_{2m-2}), \qquad \psi_e \in \Pi^1_2,$$

and suppose ξ is successful, i.e.

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y_\xi, a_0^\xi, a_1, \ldots, a_{2m-2})$$

holds. Then for every $\beta > \xi$,

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \ \mathrm{NotCoded}^{\omega_1}_\Sigma(\mathsf{U}_{\mathrm{odd}}, e, x, y_\beta, a_0^\beta, a_1, \ldots, a_{2m-2}). \tag{1}$$

If instead

$$\theta_e(x,y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \forall a_{2m-3} \psi_e(x, y, a_0, \ldots, a_{2m-3}), \qquad \psi_e \in \Sigma^1_2,$$

and ξ is successful, then for every $\beta > \xi$,

$$\forall a_1 \exists a_2 \cdots \forall a_{2m-3} \ \mathrm{Coded}^{\omega_1}_\Sigma(\mathsf{U}_{\mathrm{even}}, e, x, y_\beta, a_0^\beta, a_1, \ldots, a_{2m-3}). \tag{2}$$

(iv) *The least successful candidate is selected. In the odd-level case above, if $\xi > 0$ is least such that*

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x, y_\xi, a_0^\xi, a_1, \ldots, a_{2m-2})$$

holds, then

$$\exists a_1 \forall a_2 \cdots \forall a_{2m-2} \ \mathrm{Coded}^{\omega_1}_\Sigma(\mathsf{U}_{\mathrm{odd}}, e, x, y_\xi, a_0^\xi, a_1, \ldots, a_{2m-2}). \tag{3}$$

In the even-level case above, if $\xi > 0$ is least successful, then

$$\exists a_1 \forall a_2 \cdots \exists a_{2m-3} \ \mathrm{NotCoded}^{\omega_1}_\Sigma(\mathsf{U}_{\mathrm{even}}, e, x, y_\xi, a_0^\xi, a_1, \ldots, a_{2m-3}). \tag{4}$$

(v) *The resulting definitions uniformize the projective relations. For odd levels, let*

$$\theta_e(x,y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x,y,a_0,a_1,\ldots,a_{2m-2}), \qquad \psi_e \in \Pi^1_2.$$

Define $U^{\omega_1}_e(x,y)$ as follows. First, $U^{\omega_1}_e(x,0)$ holds if

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x,0,0,a_1,\ldots,a_{2m-2})$$

holds. If this initial clause fails, then $U^{\omega_1}_e(x,y)$ holds iff there is a real a_0 such that

$$\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \psi_e(x,y,a_0,a_1,\ldots,a_{2m-2})$$

and

$$\neg\Big[\forall a_1 \exists a_2 \cdots \exists a_{2m-2} \ \mathrm{NotCoded}^{\omega_1}_{\Sigma}(\mathsf{U}_{\mathrm{odd}}, e, x, y, a_0, a_1, \ldots, a_{2m-2})\Big].$$

Then $U^{\omega_1}_e$ is a Σ^1_{2m+1} uniformization of the relation defined by θ_e on its projection. For even levels,

$$\theta_e(x,y) \equiv \exists a_0 \forall a_1 \exists a_2 \cdots \forall a_{2m-3} \psi_e(x,y,a_0,a_1,\ldots,a_{2m-3}), \qquad \psi_e \in \Sigma^1_2,$$

the same definition is used with the parity reversed: replace $\mathsf{U}_{\mathrm{odd}}$ by $\mathsf{U}_{\mathrm{even}}$ and replace the final $\mathrm{NotCoded}^{\omega_1}_{\Sigma}$ by $\mathrm{Coded}^{\omega_1}_{\Sigma}$. The resulting predicate is a Σ^1_{2m} uniformization of the relation defined by θ_e on its projection.

Theorem 7.3. *Assume $V = L$. Let $G \subseteq \mathbb{P}_{\omega_2}$ be generic for the iteration of Subsection 7.2. Then in $L[G]$,*

$$\mathfrak{b} = \omega_1, \qquad \mathfrak{c} = \omega_2,$$

there is a Π^1_2 maximal orthogonal family of Borel probability measures on 2^ω, there are no boldface Σ^1_2 maximal orthogonal families, and UP_Σ holds.

Proof. The iteration is S-proper, so it preserves ω_1. The standard preservation theorem for the Fischer–Friedman–Törnquist countable-support ω_2 construction gives preservation of ω_2 as well. Each factor is ω^ω-bounding, and countable-support iterations of S-proper ω^ω-bounding forcings of this form remain ω^ω-bounding. Since L satisfies CH, there is in L an unbounded family

$$B \subseteq \omega^\omega \qquad \text{with} \qquad |B| = \omega_1.$$

The ω^ω-bounding preservation theorem implies that B remains unbounded in $L[G]$. Hence

$$\mathfrak{b}^{L[G]} \leq \omega_1.$$

The reverse inequality is automatic, and therefore

$$\mathfrak{b}^{L[G]} = \omega_1.$$

The continuum is ω_2. By GCH in L and the countable-support size calculation,

$$|\mathbb{P}_{\omega_2}| = \omega_2.$$

Thus

$$\mathfrak{c}^{L[G]} \leq \omega_2.$$

On the other hand, the first factor at every stage is random forcing, so the iteration adds new reals cofinally often in ω_2. Therefore

$$\mathfrak{c}^{L[G]} \geq \omega_2.$$

Consequently

$$\mathfrak{c}^{L[G]} = \omega_2.$$

We next verify the maximal orthogonal family. At a successful MO-stage, the construction considers a non-atomic measure code x which is orthogonal to the family already built, chooses a fresh MO-block I_λ, forces with

$$\mathrm{Code}_{\omega_1}(\lambda, \mathsf{MO}, x),$$

and obtains the perfect-tree coding real $r_{\lambda,\mathsf{MO},x}$. It then inserts

$$\mu_{G(x, r_{\lambda,\mathsf{MO},x})}.$$

By the Fischer–Törnquist coding lemma,

$$\mu_{G(x, r_{\lambda,\mathsf{MO},x})} \approx \mu_x,$$

and the inserted measure code FT-decodes to $r_{\lambda,\mathsf{MO},x}$.

The projective definition is the same as in Lemma 5.1, with countable ω_2-suitable models and the ω_1-coding predicate in place of the large-continuum predicate. Namely, a non-atomic measure code ρ belongs to the family iff every countable ω_2-suitable transitive model M containing ρ sees that ρ FT-decodes to a real r, that r locally MO-decodes a non-atomic measure code x through an internal ω_1-coding block, and that

$$\rho = G(x, r).$$

Once M is fixed, all these assertions are first-order over the coded model, apart from the recursive FT-decoding and the recursive equation with G. Therefore the definition is Π^1_2.

Every intentionally inserted measure satisfies this condition. Conversely, if a measure code satisfies it, then the local MO-block seen by the countable suitable models cannot come from a protected coordinate. Lemma 7.1, together with the stationarity preservation of the interleaved random factors, implies that the block must have been intentionally used at a successful MO-stage. At that stage the construction inserted precisely the code $G(x, r)$. Thus the Π^1_2 definition is exact.

Orthogonality follows from the success criterion at the construction stages: when a measure code x is accepted, μ_x is orthogonal to all earlier members, and the inserted measure is equivalent to μ_x. Maximality among non-atomic measures follows from the cofinal bookkeeping: every name for a non-atomic measure is considered, and if it is still orthogonal to the family constructed so far, an equivalent measure is inserted. Finally, adjoining all point masses gives a maximal orthogonal family in $\mathcal{P}(2^\omega)$, and the set of point masses is Borel, so the resulting family remains Π^1_2.

There are no boldface Σ^1_2 maximal orthogonal families. Let $a \in L[G] \cap 2^\omega$. Every real name in this countable-support iteration is supported on fewer than ω_2 many coordinates, and hence on a bounded set of coordinates. Thus there is some $\alpha < \omega_2$ such that

$$a \in L[G_\alpha].$$

Choose $\beta > \alpha$. The first factor at stage β adds a random real over $L[G_\alpha]$. By the random-preservation property of the remaining tail, this real remains random over $L[G_\alpha]$, and hence over $L[a]$, in the final extension. Proposition 2.3 therefore rules out a $\Sigma^1_2(a)$ maximal orthogonal family. Since a was arbitrary, there are no boldface Σ^1_2 maximal orthogonal families.

It remains to verify UP_Σ. The case $n = 2$ is Kondo's theorem. For $n \geq 3$, let

$$A \subseteq 2^\omega \times 2^\omega$$

be a Σ^1_n relation in the final model, say defined by a projective formula with Gödel number e. The Σ-bookkeeping presents every finite request needed for the corresponding odd or even copying construction cofinally often. Lemma 7.2 gives the coherent compression of earlier candidates, the exclusion of all candidates after the least successful one, and the positive selection of the least successful candidate. Therefore the predicate $U^{\omega_1}_e$ supplied by Lemma 7.2(v) is a Σ^1_n uniformization of A on its projection. Hence every Σ^1_n relation admits a Σ^1_n uniformization, for every $n \geq 2$.

The MO- and Σ-parts do not interfere. They use disjoint reservoirs Λ_{MO} and Λ_Σ, and the tag maps ι_{MO} and ι_Σ have disjoint ranges. Thus a real written for the maximal-orthogonal-family construction cannot be decoded as a Σ-uniformization code, and conversely. This completes the proof. □

Theorem 1.2 is precisely Theorem 7.3.

8 Open questions

We close with several questions suggested by the preceding constructions.

The first question is due to Fischer and Törnquist. They proved that if there is a Cohen real over L, then there is no Π^1_1 maximal orthogonal family, and asked whether the converse obstruction is exact [FT10, Question 4.3].

Question 1 (Fischer–Törnquist). If there is a Π^1_1 maximal orthogonal family of Borel probability measures on 2^ω, must all reals be constructible?

Equivalently, one may ask whether the existence of a coanalytic maximal orthogonal family is already a global L-phenomenon. The relativized form is also natural: if there is a boldface Π^1_1 maximal orthogonal family, must the real line be contained in some $L[a]$?

The results of Fischer–Friedman–Törnquist and of the present paper show that the next level behaves differently: it is consistent that there is a Π^1_2 maximal orthogonal family but no boldface Σ^1_2 maximal orthogonal family [FFT12]. This leads to the higher-level exactness problem.

Question 2. Let $n \geq 3$. Is it consistent that there is a Π^1_n maximal orthogonal family of Borel probability measures on 2^ω, but there is no boldface Σ^1_n maximal orthogonal family?

More generally, one can ask whether the exact separation

$$\exists \Pi^1_n \text{ m.o. family} \qquad \text{and} \qquad \nexists \boldsymbol{\Sigma}^1_n \text{ m.o. family}$$

can be obtained together with corresponding projective uniformization or regularity principles at the same level.

The known obstructions also leave open a finer forcing question. At the coanalytic level, Fischer and Törnquist use Cohen reals. At the next level, Fischer–Friedman–Törnquist use Cohen or random reals to rule out Σ^1_2 maximal orthogonal families.

Question 3. Which classical real-adding forcings destroy Π^1_1 maximal orthogonal families? For example, does the existence of a random real, a Sacks real, or a Miller real over L imply that there is no Π^1_1 maximal orthogonal family?

Finally, all presently known constructions of projective maximal orthogonal families at the Π^1_2 level use a strong projective coding apparatus and produce a Δ^1_3 wellorder of the reals.

Question 4. Can there be a Π^1_2 maximal orthogonal family in a model with no Δ^1_3 wellorder of the reals? More generally, what projective wellordering or choice principles are implied by the existence of a Π^1_2 maximal orthogonal family?

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